MELLIN-TYPE CONVOLUTION OPERATORS FROM PAST TO PRESENT

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I certify that in my opinion the thesis submitted by Dilshad Qasim Hamza HASO titled "MELLIN-TYPE CONVOLUTION OPERATORS FROM PAST TO PRESENT" is fully adequate in scope and in quality as a thesis for the degree of Master of Science.

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Dilshad Qasim Hamza HASO

ABSTRACT

M. Sc. Thesis

MELLIN-TYPE CONVOLUTION OPERATORS FROM PAST TO PRESENT

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The Department of Mathematics

Thesis Advisor:
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This thesis consists of five parts. The first part of this thesis includes introductory information. The second part includes the theoretical background consisting of some important definitions and theorems. In the third part, Mellin transform, inverse Mellin transform, Fourier transform and inverse Fourier transform are discussed. Also, Mellin-type linear convolution operators are examined. In the fourth part, an overview of the literature concerning Mellin-type nonlinear operators is given. Then, a pointwise convergence result concerning generalized Mellin $m-p-$Lebesgue points of integrable functions is proved. The last part is devoted to giving concluding remarks.

Key Words : Haar measure, Mellin transform, Mellin-type convolution, m-singular integrals, pointwise convergence.

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ÖZET

Yüksek Lisans Tezi

GEÇMİŞTEN GÜNÜMÜZE MELLIN TİPİ KONVOLÜSYON OPERATÖRLER

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Bu tez beş bölümden oluşmaktadır. Tezin ilk bölümü giriş bilgilerini içermektedir. İkinci bölüm, bazı önemli tanım ve teoremlerden oluşan teorik arka planı oluşturmaktadır. Üçüncü bölümde Mellin dönüşümü, ters Mellin dönüşümü, Fourier dönüşümü ve ters Fourier dönüşümü ele alınmıştır. Ayrıca, Mellin tipi lineer konvolüsyon operatörler incelenmiştir. Dördüncü bölümde, Mellin tipi nonlineer operatörler ile ilgili literatüre genel bir bakış verilmiştir. Daha sonra, integrallenebilir fonksiyonların genelleştirilmiş Mellin $m-p-$Lebesgue noktalarına ilişkin bir noktasal yakınsama sonucu kanıtlanmıştır. Son bölüm, sonuç açıklamalarına ayrılmıştır.

Anahtar Kelimeler : Haar ölçüsü, Mellin dönüşümü, Mellin tipi konvolüsyon, m-singüler integral, noktasal yakınsama.

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SYMBOLS

$\mathbb{F}$: Field consisting of real or complex numbers

$\mathbb{C}$: The set of all complex numbers

$\mathbb{R}$: The set of all real numbers

$\mathbb{R}_+$: The set of all positive real numbers

$\mathbb{N}$: The set of all positive integers

$\mathbb{Q}$: The set of all rational numbers

$\mathbb{R}^n$: n −dimensional Euclidean space

Γ : Gamma function

B : Beta function

$f_{\mathcal{M}}$: Mellin transform

$f_{\mathcal{F}}$: Fourier transform

$\langle \alpha, \beta \rangle$: Fundamental strip

$Re[s]$: Real part of complex number s

ABBREVIATIONS

resp. : respectively

i.e. : that is or in other words

PART 1

INTRODUCTION

Mellin transforms and Mellin-type convolution operators, named after the famous mathematician Hjalmar Mellin, have been studied for many years. Lindelöf (1933) presented very inclusive information about Hjalmar Mellin and his contributions. Mamedov (1991) and Butzer and Jansche (1997) presented very important results in this respect. Mellin-type operators have also been studied extensively in approximation theory. Some studies centered on approximation by linear and nonlinear operators can be given as (Butzer and Jansche, 1998; Bardaro and Mantellini, 2006; Bardaro and Mantellini, 2007; Bardaro and Mantellini, 2011; Bardaro et al., 2013; Angeloni and Vinti 2014; Angeloni and Vinti, 2015; Fard and Zainuddin, 2016). Also, detailed information about Mellin-type nonlinear operators with some historical background can be found in (Bardaro et al., 2003).

In this thesis, although it is not possible to examine all studies, we present a small-scale study. Then, we prove a pointwise convergence result concerning generalized Mellin $m - p -$Lebesgue points of integrable functions.

PART 2

THEORETICAL AND CONCEPTUAL BACKGROUND

In this part of the thesis, well-known theorems and definitions of notions are recalled from some reference works. The information covered in this part is given for the convenience of the reader in the continuing parts.

2.1. PRELIMINARIES

Definition 2.1.1. Let Λ be an arbitrary set and $\lambda \in \Lambda$. Then, a set $(S_\lambda)_{\lambda\in\Lambda} = \{S_\lambda : \lambda \in \Lambda\}$ is called an **indexed set**. In this regard, Λ is called an **index set** (Mucuk, 2010).

Example 2.1.1. Consider the set $\{2022, 2029, 2048, \ldots\}$ which equals $\{1^3 + 2021, 2^3 + 2021, 3^3 + 2021, \ldots, \lambda^3 + 2021, \ldots\}$. This set is an indexed set with respect to $\mathbb{N}$ such that $(S_\lambda)_{\lambda\in\mathbb{N}} = (\lambda^3 + 2021)_{\lambda\in\mathbb{N}}$ with $\Lambda = \mathbb{N}$.

Definition 2.1.2. A set in which the members are sets is called a **class**. A set in which the members are classes is called a **family** (Mucuk, 2010).

Definition 2.1.3. A function $\circledast : G \times G \to G$ defined on a non-empty set G is called a **binary operation** on G. A set G with a binary operation defined on it forms an **algebraic structure** denoted by $(G, \circledast)$ (Bayraktar, 2006).

Definition 2.1.4. Let $(G, \circledast)$ be an algebraic structure. If:

- for all $a, b, c \in G$, the equality $a \circledast (b \circledast c) = (a \circledast b) \circledast c$,
- for all $a \in G$, there exists $e \in G$ such that $a \circledast e = e \circledast a = a$;
- for each $a \in G$ there exists $a^{-1} \in G$ such that $a \circledast a^{-1} = a^{-1} \circledast a = e$.

hold there, then G is called a **group** with respect to an algebraic operation $\circledast$. If, in addition, for all $a, b \in G$ the equality $a \circledast b = b \circledast a$ hold, then $(G, \circledast)$ is called an **Abelian** (commutative) group (Bayraktar, 2006).

Example 2.1.2. It is known that with regards to the usually utilized addition operation, the set of all real numbers $\mathbb{R}$ forms an Abelian group. In relation to usually utilized multiplication, the set of all positive real numbers $\mathbb{R}_+$ also forms an Abelian group. On the other hand, due to the absence of the inverse of zero, $\mathbb{R}$ is not a group in relation to the usually utilized multiplication process. For some detailed explanations and further examples; see (Bayraktar, 2006).

Definition 2.1.5. Let $\mathbb{F}$ be a field consisting of real or complex numbers and $\mathcal{V}$ be a non-empty set. If the following conditions with respect to two algebraic operations given as addition "$\oplus$" and scalar multiplication "$\circledast$":

$\mathcal{V}$ verifies the following properties with respect to addition "$\oplus$":

- for all $x, y \in \mathcal{V}$ one has $x \oplus y \in \mathcal{V}$;
- for all $x, y \in \mathcal{V}$, one has $x \oplus y = y \oplus x$;
- for all $x, y, z \in \mathcal{V}$, one has $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- there exists a zero element "**O**" in $\mathcal{V}$ such that for all $x \in \mathcal{V}$, one has $x \oplus \mathbf{O} = \mathbf{O} \oplus x = x$;
- for each $x \in \mathcal{V}$ there exists $x^{-1} \in \mathcal{V}$ with $x^{-1} = -x$ with respect to addition such that $x \oplus (-x) = (-x) \oplus x = \mathbf{O}$ holds for all $x \in \mathcal{V}$;

$\mathcal{V}$ verifies the following properties with respect to scalar multiplication "$\circledast$":

- ➢ $\alpha \circledast x \in \mathcal{V}$ for all $x \in \mathcal{V}$ and $\alpha \in \mathbb{F}$;
- ➢ $\alpha \circledast (x \oplus y) = (\alpha \circledast x) \oplus (\alpha \circledast y)$ and
  $(\alpha \oplus \beta) \circledast x = (\alpha \circledast x) \oplus (\beta \circledast x)$ for all $x, y \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{F}$;
- ➢ $(\alpha \circledast \beta) \circledast x = \alpha \circledast (\beta \circledast x)$ for all $x \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{F}$;
- ➢ $\mathbf{1} \circledast x = x$ for all $x \in \mathcal{V}$, where $\mathbf{1}$ is an identity element of $\mathbb{F}$;

hold, then $\mathcal{V}$ is called a **vector space** over $\mathbb{F}$. Here, the members of $\mathcal{V}$ and $\mathbb{F}$ are respectively called **vectors** and **scalars** (Bayraktar, 2017).

Example 2.1.3. The set of all real numbers $\mathbb{R}$ and the set of all complex number $\mathbb{C}$ are well-known vector spaces with respect to their usual addition and multiplication operations, for some details and further examples, (Bayraktar, 2017) is recommended.

Definition 2.1.6. Let X be a set. A real-valued function $d(x,y)$ defined on $X \times X$ satisfying the following properties:

- $d(x,y) \geq 0$ holds for all $x, y \in X$ with $d(x,y) = 0 \Leftrightarrow x = y$;
- $d(x,y) = d(y,x)$ holds for all $x, y \in X$;
- $d(x,y) \leq d(x,z) + d(z,y)$ holds for all $x, y \in X$;

is called a **metric** (or a distance function**)** on X. A pair (X,d) is called a **metric space** (Kolmogorov and Fomin, 1975).

Example 2.1.4. Let $X = \mathbb{R}$. An absolute value function defined on $\mathbb{R} \times \mathbb{R}$ with $d(x,y) = |x - y|$ (Euclidean distance) is a well-known metric for $\mathbb{R}$ (Kolmogorov and Fomin, 1975).

Definition 2.1.7. Let (X_1, d_1) and (X_2, d_2) be two metric spaces. Further, let g be a function from X_1 to X_2. If for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_1(x_0, x) < \delta \Rightarrow d_2\big(g(x_0), g(x)\big) < \varepsilon$ then g is continuous at $x_0 \in X_1$. If g is continuous at each $x_0 \in X_1$, then g is **continuous** on X_1 (Folland, 1999).

Example 2.1.5. Polynomials and an exponential function e^x defined on $\mathbb{R}$ are well-known continuous functions with respect to usual Euclidean distance $d(x,y) = |x - y|$ with $(x,y) \in \mathbb{R} \times \mathbb{R}$.

Remark 2.1.1. In this thesis, we will consider the usual Euclidean distance as a metric unless otherwise specified.

Definition 2.1.8. A function $g: \mathbb{R} \rightarrow \mathbb{C}$ is said to be **Lipschitz** continuous if there is a constant C such that

$|g(\lambda) - g(y)| \leq C\ |\lambda - \xi|$

holds for all $\lambda, \xi \in \mathbb{R}$ (Folland, 1999).

Example 2.1.6. Let $g_1(\lambda) = \cos(\lambda)$ with $g_1: \mathbb{R} \rightarrow [-1,1]$ and $g_2(\lambda) = |\lambda|$ with $g_2: \mathbb{R} \rightarrow \mathbb{R}_+ \cup \{0\}$. These are well-known Lipschitz continuous functions since

$|\cos(\lambda) - \cos(\xi)| \leq\ C_1|\lambda - \xi|$ with $C_1 = 1$

and

$|\ |\lambda| - |\xi|\ | \leq\ C_2|\lambda - \xi|$ with $C_2 = 1$

for all $\lambda, \xi \in \mathbb{R}$.

The first inequality follows from the fact that:

$$
\begin{aligned}
|\cos(\lambda) - \cos(\xi)| &= 2\left|\sin\left(\frac{\lambda+\xi}{2}\right) \cdot \sin\left(\frac{\lambda-\xi}{2}\right)\right| \\
&\leq 1 \cdot 2 \cdot \left|\frac{\lambda-\xi}{2}\right| \cdot \left|\sin\left(\frac{\lambda+\xi}{2}\right)\right| \\
&\leq |\lambda - \xi|,
\end{aligned}
$$

where $\left|\sin\left(\frac{\lambda+\xi}{2}\right)\right| \leq 1$ and $\left|\sin\left(\frac{\lambda-\xi}{2}\right)\right| \leq \frac{1}{2}\ |\lambda - \xi|$, for all $\lambda, \xi \in \mathbb{R}$.

The second one is clear by the well-known triangle inequality given by (see, for example, (Balcı, 2008)):

$|\ |\lambda| - |\xi|\ | \leq\ |\lambda - \xi| \leq |\lambda| + |\xi|$, where $\lambda, \xi \in \mathbb{R}$.

Definition 2.1.9. Let X be a non-empty set in $[-\infty, +\infty]$. The least upper bound of X is called **supremum** of X, and it is denoted by $\sup X$. Similarly, the greatest lower bound of X is called **infimum** of X, and it is denoted by $\inf X$. Also, for a function g defined on X, the symbols $\sup_{u \in X} g(u)$ and $\inf_{u \in X} g(u)$ are respectively used for supremum and infimum values of g (Rudin, 1987).

Example 2.1.7. Let $X = (2020, 2021)$. Here, $\sup X = 2021$ and $\inf X = 2020$ for all $x \in X$.

Definition 2.1.10. Let I be an arbitrary interval in $\mathbb{R}$ and g be a real-valued function defined on this interval. If there holds the following inequality:

$g(\alpha x_1 + (1-\alpha)x_2) \leq \alpha g(x_1) + (1-\alpha)g(x_2)$ for all $x_1, x_2 \in I$ and $\alpha \in [0,1]$, then g is **convex** on I. Similarly, if there holds the inequality:

$g(\alpha x_1 + (1-\alpha)x_2) \geq \alpha g(x_1) + (1-\alpha)g(x_2)$ for all $x_1, x_2 \in I$ and $\alpha \in [0,1]$, then g is **concave** on I (Dernek, 2003).

Example 2.1.8. Let $f, g: \mathbb{R} \to \mathbb{R}$ be such that $f(x) = x^2$ and $g(x) = -x^2$ then $f(x)$ is convex on $\mathbb{R}$ and $g(x)$ is concave on $\mathbb{R}$.

Definition 2.1.11. Let X be a vector space. A function g on X satisfying the following properties:

- $g(x) \geq 0$ holds for all $x \in X$ with $g(x) = 0 \Leftrightarrow x = 0$;
- $g(\alpha x) = |\alpha| g(x)$ holds for all $x \in X$ and scalar $\alpha \in \mathbb{F}$ ($\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$);
- $g(x + y) \leq g(x) + g(y)$ holds for all $x, y \in X$;

is called a **norm** on X. A vector space X with a norm $g(x) = \|x\|$ is called a **normed space** (Kolmogorov and Fomin, 1975).

Example 2.1.9. $\mathbb{R}$ is a normed space by taking $\|u\| = |u|, \quad \forall u \in \mathbb{R}$ (Kolmogorov and Fomin, 1975).

Now, we will give some necessary topological preliminaries.

Definition 2.1.12. Let X be a set and τ be a collection consisting of subsets of this set. If τ possesses the following properties: i.e.;

- $X, \emptyset \in \tau$;
- if $\{S_i\}_{i=1}^{n} \in \tau$, then $\cap_{i=1}^{n} S_i \in \tau$, where $n \in \mathbb{N}$;
- if for an arbitrarily selected collection of elements $\{S_\alpha\}$ which is finite, countable or uncountable, then $\cup_{\alpha \in I} S_\alpha \in \tau$, where I is an index set;

then τ is called a **topology** on X. (X, τ), a set X with a topology defined on it, is called a **topological space** and its elements are called **open sets** of X (Rudin, 1987).

Example 2.1.10. Let $X = \{2019,\ 2020,\ 2021\}$ be given. Considering the sets given by $\tau_1 = \{X,\ \emptyset,\ \{2019\},\ \{2020\},\ \{2019,\ 2020\}\}$ and $\tau_2 = \{X,\ \emptyset,\ \{2019,\ 2020\},$ $\{2019,\ 2021\}\}$, one observes that τ_1 is a topology on X. On the contrary, since $\{2019,\ 2020\} \cap \{2019,\ 2021\} = \{2019\} \notin \tau_2,\ \tau_2$ is not a topology on X.

Definition 2.1.13 Let $S \subset \mathbb{R}$. If for every $x \in S$ there exists $\varepsilon = \varepsilon_x > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset S$, then S is an **open set** in $\mathbb{R}$. If for a given set $S \subset X$ the set $S^C = X \backslash S$ is open in X, then S is called a **closed set.** A class of open sets $\mathcal{U}$ in $\mathbb{R}$ is a topology on $\mathbb{R}$, and it is called the **usual topology** on $\mathbb{R}$ (Mucuk, 2010).

Definition 2.1.14. Let (X, τ) be a topological space, $S \subseteq X$, and s denote an element of S. If there is an open set $O \in \tau$ with $s \in O \subseteq S$, then a set S is called a **neighbourhood** of s (Mucuk, 2010).

Example 2.1.11. For $x = 0$ and $\varepsilon = \frac{1}{4}$, one has $(x - \varepsilon, x + \varepsilon) = \left(0 - \frac{1}{4}, 0 + \frac{1}{4}\right) = \left(-\frac{1}{4}, \frac{1}{4}\right) = S$. Here, S is a neighbourhood of $x = 0$ with respect to $\varepsilon = \frac{1}{4}$ in view of usual topology on $\mathbb{R}$.

Definition 2.1.15. Let (X, τ) be a topological space, $S \subseteq X$ and $x \in X$. If for every open neighbourhood O of $x \in X$ the set $(O \backslash \{x\}) \cap S$ is non-empty, then a point $x \in X$ is called an **accumulation point** of S (Mucuk, 2010).

Example 2.1.12. Let $S = (2020, 2021)$. Considering $\mathbb{R}$ with its usual topology, we see that all elements of S together with {2020, 2021} are accumulation points of the set S.

Proposition 2.1.1. Every subset of Euclidean space $\mathbb{R}^n$ which is closed and bounded is compact (Folland, 1999).

Example 2.1.13. It is well-known that any closed and bounded interval $[a, b]$ in $\mathbb{R}$ is compact.

Definition 2.1.16. Let (X, τ) be a topological space in whose every point possesses a compact neighbourhood is called a **locally compact topological space** (Folland, 1999).

Example 2.1.14. The set $\mathbb{R}$ is local compact with respect to the usual topology. But, since none of the elements of $\mathbb{Q}$ have a compact neighbourhood, $\mathbb{Q}$ is not locally compact (Mucuk, 2010).

Definition 2.1.17. Let (X, τ) be a topological space. If X possesses the property given as if $u \neq v$, then there are open sets U and V satisfying $U \cap V = \emptyset$, where $u \in U$ and $v \in V$, then X is called **Hausdorff space** (Folland, 1999).

Definition 2.1.18. A group $(G, \circledast)$ with a topology defined on it such that the following group operations: $(a, b) \to a \circledast b$ and $a \to a^{-1}$ are respectively continuous from $G \times G \to G$, and $G \to G$ is called a **topological group.** In particular, a topological group on which the topology is locally compact and Hausdorff is called a **locally compact group** (Folland, 1999).

Example 2.1.15. $\mathbb{R}_+$ with usual multiplication and usual topology came from $\mathbb{R}$ is a well-known (multiplicative) topological group.

Now, we will present some necessary measure theoretical preliminaries.

Definition 2.1.19. Let X be a set. Further, let $\mathcal{J}$ denote a collection consisting of subsets of this set. If $\mathcal{J}$ possesses the following properties:

- $X \in \mathcal{J}$;
- if a set $S \in \mathcal{J}$, then $S^c \in \mathcal{J}$, where S^c denotes complement of S in X ($S^c = X \setminus S$);
- if $\{S_n\}_{n \in \mathbb{N}} \in \mathcal{J}$, then $\cup_{n=1}^{\infty} S_n \in \mathcal{J}$;

then $\mathcal{J}$ is called $\boldsymbol{\sigma}-$**algebra** on X. A pair $(X, \mathcal{J})$ is called a **measurable space,** and elements of $\mathcal{J}$ are called **measurable sets** on X (Rudin, 1987).

Definition 2.1.20. Let $(X, \mathcal{J})$ be a measurable space. The function $\mu: \mathcal{J} \rightarrow [0, +\infty]$ is called a **measure** if for each infinite sequence (S_n) consisting of disjoint elements of $\mathcal{J}$ there holds $\mu(\cup_{n=1}^{\infty} S_n) = \sum_{n=1}^{\infty} \mu(S_n)$ and $\mu(\emptyset) = 0$. $(X, \mathcal{J}, \mu)$ is called a **measure space** (Cohn, 1980).

Definition 2.1.21. Let (X, τ) be a topological space. The Borel σ-algebra, which is denoted by $B(X)$ is the σ-algebra generated by the family of open sets (respectively closed sets) in X. The elements of $B(X)$ are called **Borel sets**. In particular, if $X = \mathbb{R}$, then Borel σ-algebra on $\mathbb{R}$ is obtained (Folland, 1999).

Proposition 2.1.2. Open and closed intervals, semi-open (respectively semi-closed) intervals and semi-infinite intervals in $\mathbb{R}$ generate $B(\mathbb{R})$ separately (Folland, 1999).

Example 2.1.16. Let $B(X) = B(\mathbb{R})$. The intervals $(0,1)$, $(0, +\infty)$ and $(0,4]$ are some elements of $B(\mathbb{R})$.

Definition 2.1.22. Let $P(\mathbb{R})$ denote the set consisting of all subsets of $\mathbb{R}$. A function $\mu^*: P(\mathbb{R}) \rightarrow [0, +\infty]$ defined by

$$\mu^*(S) = \inf \left\{ \sum_i (c_i - d_i) : \{(c_i, d_i)\} \in \theta_S \right\}$$

for each subset $S \subset \mathbb{R}$ satisfying $S \subset \cup_{i=1}^{\infty} (c_i, d_i)$, where θ_S is a set of all infinite sequences $\{(c_i, d_i)\}$ of open bounded intervals, that is, $c_i, d_i \in \mathbb{R}$ with $c_i < d_i$ for each i, is called **Lebesgue outer measure** (Cohn, 1980)**.**

In particular, if μ^* is considered on $B(\mathbb{R})$, then it is called **Lebesgue measure** shown as μ^{**} and domain of it will be called a class of **Lebesgue measurable sets** (Folland, 1999).

Proposition 2.1.3. Using the Lebesgue (outer) measure, one evaluates the measure of each interval $I \subset \mathbb{R}$ as its length (Cohn, 1980; Folland, 1999).

For example, if $I = (2019, 2021)$, then $\mu^*(I) = 2021 - 2019 = 2 = \mu^{**}(I)$.

Definition 2.1.23. A Borel measure μ on a Hausdorff topological space is a measure such that $\mu: B(X) \to [0, +\infty]$. A measure defined on $B(X)$ satisfying:

- $\mu(K)$ is finite for each compact set $K \subset X$;
- $\mu(S) = \inf\{\mu(U): S \subset U$ with U being open$\}$ for each $S \in B(X)$;
- $\mu(U) = \sup\{\mu(K): K \subset U$ with K being compact$\}$ for each open set $U \subset X$;

is called a regular Borel measure (Cohn, 1980).

Definition 2.1.24. Let $(X, \mathcal{J})$ be a measurable space. The function $g: S \to [-\infty, +\infty]$, where $S \in \mathcal{J}$ is a subset of X, is called a **measurable function** if for each $k \in \mathbb{R}$, the set defined by $\{t \in S: g(t) > k\} \in \mathcal{J}$. If $X = \mathbb{R}$ and $\mathcal{J} = B(\mathbb{R})$, then the function is **Borel measurable function** (resp. Borel function). Also, if $\mathcal{J}$ stands for a Lebesgue measurable set on $\mathbb{R}$, then the function is measurable in the sense of Lebesgue (Cohn, 1980).

Definition 2.1.25. A measurable function g defined on a measurable space X with respect to Lebesgue measure μ^{**} satisfying $\int_X |g| d\mu < \infty$ is called a **Lebesgue integrable function** (Rudin, 1987).

Definition 2.1.26. Assume that (a, b) be an arbitrary interval in $\mathbb{R}$ whose endpoints satisfy the inequality $-\infty \leq a < b \leq +\infty$. Let $1 \leq p < \infty$ and g be a Lebesgue measurable function defined on this interval. Space $L^p(a, b)$, the space of functions whose p-th power is Lebesgue integrable on (a, b), consists of the functions equipped with the norm

$$\|g\|_{L^p(a,b)} = \left(\int_a^b |g(x)|^p \, dx \right)^{1/p}$$

with $\|g\|_{L^p(a,b)} < \infty$ (Hacısalihoğlu and Hacıyev, 1995).

Definition 2.1.27. Let M be an arbitrary subset of $\mathbb{R}$. A function defined by

$$\mathcal{K}_M(u) = \begin{cases} 1, & if\ u \in M, \\ 0, & if\ u \notin M. \end{cases}$$

is called a **characteristic function** of the set M (Rudin, 1987).

Now, we are ready to give the definition of Haar measure.

Definition 2.1.28. A non-zero regular left (or right) translation-invariant Borel measure μ on locally compact group G is a left (or right) **Haar measure** on G. Here, above indicated measure μ is left (or right) translation-invariant means for every $x \in G$ and $S \in B(G)$, $\mu(x \circledast S) = \mu(S)$ (or $\mu(S \circledast x) = \mu(S)$) holds. Here, for every fixed $x \in G$ the notation $x \circledast S$ (or $S \circledast x$) represents $x \circledast t$ (or $t \circledast x$) for all $t \in S$ (Cohn, 1980).

As seen below, Cohn (1980) used different notations for left and right translates and explained why he did it with reasons.

Example 2.1.17. Let G be a locally compact group with respect to operation $\circledast$. Let x^{-1} denote inverse element such that $x \circledast x^{-1} = x^{-1} \circledast x$ holds for every $x \in G$.

Denoting left translate of a function g defined on G by ${}_x g$ with ${}_x g(t) = g\left(x^{-1} \circledast t\right)$ for every $t \in G$, one has

$$\int_G {}_x g \; d\mu = \int_G g \, d\mu$$

for each Borel function g which is either integrable with respect to measure μ or non-negative, provided that μ is a left Haar measure on G. The right translate of g is defined similarly as g_x with $g_x(t) = g(t \circledast x^{-1})$ for every $t \in G$. In particular, replacing g by $\mathcal{K}_B$ of a Borel set B on G, one has

$$\int_G {}_x \mathcal{K}_B \, d\mu = \mu(x \circledast B) = \mu(B) = \int_B d\mu \,,$$

where $\mathcal{K}_B$ is the characteristic function of B (Cohn, 1980, p. 305).

Following the strategy used above, two Haar measures given in (Cohn, 1980) are examined as follows.

Example 2.1.18. It is stated in (Cohn, 1980, p. 304) that the Lebesgue measure is a left and a right Haar measure, shortly Haar measure, on topological group $\mathbb{R}$. It is well-known that $\mathbb{R}$ is an additive topological group, which is Abelian with respect to usual addition, together with its usual topology. Here, we consider the Lebesgue integral of a function g defined on $\mathbb{R}$ as $\int_{\mathbb{R}} g(t)dt$ with respect to Lebesgue measure $\mu^{**}(A) = \int_A dt,$ where A is an arbitrary Borel set on $\mathbb{R}$. In fact, following the notation of Cohn (1980) used in Example 2.1.17, it is easily seen that

$$\int_{\mathbb{R}} {}_x \mathcal{K}_A(t) \, dt = \mu^{**}(x + A) = \mu^{**}(A + x) = \int_A dt$$

holds. Let $A = [a, b]$ with $a, b \in \mathbb{R}$ satisfying $a < b$. In particular, taking $A = [a, b]$ and making the variable change $x + u = t$ with $du = dt$ gives the following result for the left translate of $\mathcal{K}_A$:

$$\int_{\mathbb{R}} {}_x\mathcal{K}_A(t)\, dt = \int_{\mathbb{R}} \mathcal{K}_A(-x + t)\, dt = \int_{\mathbb{R}} \mathcal{K}_A(u)\, du = \int_a^b du = b - a = \mu^{**}(A).$$

Similarly, for the right translate of $\mathcal{K}_B$, one has

$$\int_{\mathbb{R}} \mathcal{K}_{A_x}(t)\, dt = \int_{\mathbb{R}} \mathcal{K}_A(t - x)\, dt = \int_{\mathbb{R}} \mathcal{K}_A(u)\, du = \int_a^b du = b - a = \mu^{**}(A).$$

The following example has particular importance in view of the variable change operations in Mellin convolutions.

Example 2.1.19. It is stated in (Cohn, 1980, p. 311) that the formula given as

$$\mu(A) = \int_A \frac{1}{t}\, dt$$

defines a Haar measure on multiplicative topological group $\mathbb{R}_+$, which is Abelian with respect to usual multiplication, with the usual topology inherited from $\mathbb{R}$. Here, dt refers to usual Lebesgue measure μ^{**} and A stands for an arbitrary Borel set on $\mathbb{R}_+$.

In fact, following the notation of Cohn (1980) used in Example 2.1.17 and using variable change given by $t = xu$ with $dt = xdu$, one has

$$\int_{\mathbb{R}_+} {}_x\mathcal{K}_A(t) \frac{1}{t} dt = \int_{\mathbb{R}_+} \mathcal{K}_A\left(\frac{1}{x} t\right) \frac{1}{t} dt = \int_{\mathbb{R}_+} \mathcal{K}_A(u) \frac{1}{u} du = \int_A \frac{1}{u} du = \mu(A)\,.$$

A similar result can be obtained for the right translate.

Let $A = [a, b]$ with $a, b \in \mathbb{R}_+$ satisfying $a < b$. Then, one has
$\mu(A) = \mu([a, b]) = \ln(b) - \ln(a) = \ln(ba^{-1})$.

Theorem 2.1.1. Assume that (a, b) be an arbitrary interval in $\mathbb{R}$ whose endpoints satisfy the inequality $-\infty \leq a < b \leq +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $1 < p, q < +\infty$. If $f \in L^p(a, b)$ and $g \in L^q(a, b)$, then the inequality:

$$\|fg\|_{L^1(a,b)} \leq \|f\|_{L^p(a,b)} \|g\|_{L^q(a,b)}$$

holds. This inequality is known as **Hölder's inequality** (Hacısalihoğlu and Hacıyev, 1995).

Theorem 2.1.2. Let $g(u, v)$ with $u, v \in \mathbb{R}$ be a bivariate (complex-valued) function defined and measurable on $\mathbb{R}^2$. Then, $g \in L^1(\mathbb{R}^2)$ and

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(u, v)\, du\, dv = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} g(u, v)\, dv \right) du = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} g(u, v)\, du \right) dv$$

provided that $\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} |g(u, v)|\, dv \right) du$ finitely exists (Butzer and Nessel, 1971).
Note that Theorem 2.1.2 is the second part of **Fubini's** theorem given in (Butzer and Nessel, 1971).

Theorem 2.1.3. Let $g(u, v)$ with $u, v \in \mathbb{R}$ be a bivariate (complex-valued) function defined and measurable on $\mathbb{R}^2$ and $1 \leq p < +\infty$. Then,

$$\left\| \int_{-\infty}^{+\infty} g(\cdot, v) dv \right\|_{L^p(\mathbb{R})} \leq \int_{-\infty}^{+\infty} \|g(\cdot, v)\|_{L^p(\mathbb{R})}\, dv$$

provided that $\|g(\cdot, v)\|_{L^p(\mathbb{R})} \in L^1(\mathbb{R})$. This inequality is called **Hölder-Minkowski** inequality (Butzer and Nessel, 1971).

Definition 2.1.29. A point $y \in \mathbb{R}$ is called a **Lebesgue point** of g if the following equality:

$$\lim_{k \to 0^+} \int_{y}^{y+k} |g(t) - g(y)|\, dt = 0$$

holds at $y \in \mathbb{R}$. Here, g is integrable on the sufficiently large domain (Natanson, 1964).

Example 2.1.20. Let $g: \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ with $g(t) = e^t$. This function is integrable on any bounded interval of $\mathbb{R}_+ \cup \{0\}$. Further, let $y = 0$. Now, we show that this point is Lebesgue point of g as follows:

$$\begin{aligned}
\lim_{k \to 0^+} \frac{1}{k} \int_0^k |e^{t+y} - e^y|\, dt &= \lim_{k \to 0^+} \frac{1}{k} \int_0^k (e^t - 1)\, dt \\
&= \lim_{k \to 0^+} \frac{1}{k}\ [e^t - t|_0^k \\
&= \lim_{k \to 0^+} \frac{1}{k}\ [(e^k - k) - (e^0 - 0)] \\
&= \lim_{k \to 0^+} \frac{(e^k - k - 1)}{k}.
\end{aligned}$$

Here, we use well-known L'Hospital's theorem such that

$$\begin{aligned}
\lim_{k \to 0^+} \frac{(e^k - 1 - 0)}{1} &= \frac{e^0 - 1}{1} \\
&= \frac{1 - 1}{1} \\
&= 0.
\end{aligned}$$

Definition 2.1.30. It is said that a property ***P*** holds almost everywhere on a set A if ***P*** holds everywhere on A except a set $S \subset A$ which is a set of measure zero (Natanson, 1964).

Definition 2.1.31. Let $g: A \subset \mathbb{R} \to \mathbb{R}$. If for all $u_1, u_2 \in A$ with $u_1 \leq u_2$ there holds $g(u_1) \leq g(u_2)$, then g is called **non-decreasing** on A. Similarly, if for all $u_1, u_2 \in A$ with $u_1 \leq u_2$ there holds $g(u_1) \geq g(u_2)$, then g is called **non-increasing** on A. A function g which is either non-increasing or non-decreasing on its domain, is called **monotonic function** (Kolmogorov and Fomin, 1975).

Example 2.1.21. Let $g: \mathbb{R} \to \mathbb{R}$ be given by $g(u) = u^4$. This function is non-decreasing on $[0, \infty)$ and non-increasing on $(-\infty, 0]$. This function is not monotonic since it is neither non-increasing nor non-decreasing in its domain.

Now, we present some necessary information about the functions of bounded variation.

Definition 2.1.32. Assume that h is a real-valued function defined on $[c, d]$, where $[c, d] \subset \mathbb{R}$ with $c < d$. Let $p = \{t_0, \ldots, t_n\}$ be any partition of the interval $[c, d]$ and **P** be the set consisting of all (finite) partitions of this interval. Total variation of h over $[c, d]$

$$\bigvee_c^d (h) = \sup_{p \in \mathbf{P}} \sum_{k=1}^{n} |h(t_k) - h(t_{k-1})|$$

is the number in $[0, +\infty]$. If $\bigvee_c^d (h)$ has a finite value which is independent of the selection of the partition, then h is said to be of **bounded variation** over $[c, d]$ (Kolmogorov and Fomin, 1975).

Theorem 2.1.4. If a function h defined on $[c, d]$, where $[c, d] \subset \mathbb{R}$ with $c < d$, is monotonic, then this function possesses a finite derivative h' almost everywhere on $[c, d]$. This theorem is called **Lebesgue's theorem** (Kolmogorov and Fomin, 1975).

Theorem 2.1.5. A function which is of bounded variation on $[c, d]$, where $[c, d] \subset \mathbb{R}$ with $c < d$, can be written as the difference of two non-decreasing functions on this interval (Kolmogorov and Fomin, 1975).

Corollary 2.1.1. A function which is of bounded variation on $[c, d]$, where $[c, d] \subset \mathbb{R}$ with $c < d$, possesses finite derivative almost everywhere on this interval (Kolmogorov and Fomin, 1975).

Corollary 2.1.2. If the function h is integrable in the sense of Lebesgue on $[c, d] \subset \mathbb{R}$ with $c < d$, then the indefinite integral function defined by

$$\mathcal{H}(t) = \int_c^t h(u)du,$$

where $t \in [c, d]$, is of bounded variation on $[c, d] \subset \mathbb{R}$ (Kolmogorov and Fomin, 1975).

Definition 2.1.33. Let $\{(c_i, d_i)\}$ be a finite sequence of disjoint and open intervals in the closed interval $\boldsymbol{I} \subset \mathbb{R}$. If $\delta > 0$ can be found for each $\epsilon > 0$ such that

$$\sum_{i=1}^{n}(d_i - c_i) < \delta \implies \sum_{i=1}^{n}|h(d_i) - h(c_i)| < \epsilon,$$

which is independent of the selection of the sequence, then the function $h: \boldsymbol{I} \to \mathbb{R}$ is **absolutely continuous** on $\boldsymbol{I} \subset \mathbb{R}$. The interval $\boldsymbol{I}$ can also be taken as $\mathbb{R}$ (Cohn, 1980).

Theorem 2.1.6. An absolute continuous function on a closed and bounded interval of $\mathbb{R}$ is of bounded variation in the same interval (Cohn, 1980).

Theorem 2.1.7. If the function h is integrable in the sense of Lebesgue on $[c, d] \subset \mathbb{R}$ with $c < d$, then the indefinite integral function defined by

$$\mathcal{H}(t) = \int_c^t h(u)du,$$

where $t \in [c, d]$, is absolutely continuous on $[c, d] \subset \mathbb{R}$ (Kolmogorov and Fomin, 1975).

The following theorem provides the conditions for transforming the Lebesgue integral into Stieltjes integral.

Theorem 2.1.8. If the function h is integrable in the sense of Lebesgue on $[c, d] \subset \mathbb{R}$ with $c < d$ and $\mathcal{H}$ is the indefinite integral of function g, which is integrable in the sense of Lebesgue on $[c, d]$, defined by

$$\mathcal{H}(t) = \int_{c}^{t} g(u)du,$$

where $t \in [c, d]$, then the equality:

$$\int_{c}^{d} h(u)g(u)du = \int_{c}^{d} h(u)d\mathcal{H}(u)$$

holds (Hobson, 1921).

Theorem 2.1.9. Let $\mathcal{H}$ and F be given functions defined on $[c, d] \subset \mathbb{R}$ with $c < d$. If one of the integrals $\int_c^d F(u)d\mathcal{H}(u)$ and $\int_c^d \mathcal{H}(u)dF(u)$ (finitely) exists, then the other one also finitely exists and the following equality:

$$\int_{c}^{d} \mathcal{H}(u)dF(u) = \mathcal{H}(d)F(d) - \mathcal{H}(c)F(c) - \int_{c}^{d} F(u)d\mathcal{H}(u)$$

holds. This formula is known as a **method of integration by parts** for Stieltjes integrals (Natanson, 1964).

Example 2.1.22. Let $h, g: [2, 3] \to \mathbb{R}$ be given by $h(u) = u^2$ and $g(u) = \frac{1}{u}$. Direct computation gives $\int_2^3 u^2 \frac{1}{u} du = \frac{5}{2}$. On the other hand, in view of $G(t) = \int_2^t \frac{1}{u} du$ and using integration by parts, we have

$$\int_2^3 u^2 dG(u) = 9\ln(3) - 9\ln(2) - \int_2^3 G(u) d(u^2)$$

$$= 9\ln(3/2) - 2\int_2^3 G(u) u du$$

$$= 9\ln(3/2) - 2\int_2^3 G(u) u du$$

$$= \int_2^3 u^2 \frac{1}{u} du$$

$$= \frac{5}{2}.$$

Definition 2.1.34. The space $\mathcal{L}^p(\mathbb{R}_+)$ consists of the measurable functions f for which $\int_0^{+\infty} |f(t)|^p \frac{1}{t} dt < \infty,$ where $t \in \mathbb{R}_+$ and $1 \le p < \infty.$ The norm in this space is defined as follows:

$$\|f\|_{\mathcal{L}^p(\mathbb{R}_+)} = \left(\int_0^{+\infty} |f(t)|^p \frac{1}{t} dt \right)^{\frac{1}{p}},$$

where $t \in \mathbb{R}_+$ and $1 \le p < +\infty$ (Mamedov, 1991). Here, the measure is as treated in Example 2.1.19. We use this norm in Parts 3 and 4 while treating Mellin-type operators.

Remark 2.1.2. Let $\mathcal{L}^1_{Loc}(\mathbb{R}_+)$ represent the space of locally integrable functions $\mathfrak{h}$ for which $\int_0^a |\mathfrak{h}(t)| \frac{1}{t} dt < +\infty,$ where $a \in \mathbb{R}_+$ and $t \in (0, a]$. Here, the measure is as treated in Example 2.1.19. Similarly, let $L^1_{Loc}(\mathbb{R}_+)$ represent the space of usual locally integrable functions $\mathfrak{h}$ for which $\int_0^a |\mathfrak{h}(t)|\, dt < +\infty$ for every choice of $a \in \mathbb{R}_+$ with $t \in (0, a]$. Here, the measure is as treated in Example 2.1.18. Considering these, Bardaro and Mantellini (2006) proved the inclusion relation given as $\mathcal{L}^1_{Loc}(\mathbb{R}_+) \subset L^1_{Loc}(\mathbb{R}_+)$ provided that $0 < a < 1$ by stating the following observation:

$$\int_0^a |\mathfrak{h}(t)|\, dt \leq \int_0^a |\mathfrak{h}(t)| \frac{1}{t} dt.$$

We may express this fact in more detailed form as follows:

$$\int_0^a |\mathfrak{h}(t)|\, dt = \int_0^a |\mathfrak{h}(t)| \frac{t}{t} dt \leq \left[\sup_{t\in(0,1)} t\right] \int_0^a |\mathfrak{h}(t)| \frac{1}{t} dt = \int_0^a |\mathfrak{h}(t)| \frac{1}{t} dt.$$

This means if $\mathfrak{h} \in \mathcal{L}^1_{Loc}(\mathbb{R}_+)$, one has $\mathfrak{h} \in L^1_{Loc}(\mathbb{R}_+)$ with $0 < a < 1$. But, the converse of this may not be true. Considering $\mathfrak{h}$ as a constant function is enough to see this. Taking $\mathfrak{h}(t) = 1$ gives indeed $\int_0^a |\mathfrak{h}(t)|\, dt = \int_0^a 1\, dt = a$ and non-existence of the integral $\int_b^a |\mathfrak{h}(t)|\, \frac{1}{t} dt$ when $b \to 0^+$. Now, one may understand why the theory developed for Lebesgue integration is used in the proofs concerning Mellin-type convolutions (see Part 4).

Remark 2.1.3. The symbols $\mathcal{L}^1_{Loc}(\mathbb{R}_+)$ and $L^1_{Loc}(\mathbb{R}_+)$ are used as in Remark 2.1.2 throughout this thesis.

Example 2.1.23. Let $f: \mathbb{R}_+ \to \mathbb{R}$ be defined by $f(t) = te^{-t}$. Now, we will show that $f \in \mathcal{L}^2(\mathbb{R}_+)$. By definition

$$\int_0^{+\infty} |f(t)|^2 \frac{1}{t} dt = \int_0^{+\infty} |t\, e^{-t}|^2 \frac{1}{t} dt$$

$$= \int_{0}^{+\infty} t\, e^{-2t}\, dt.$$

Let $u = t$ with $du = dt$ and $dv = e^{-2t}\, dt$ with $v = \frac{-1}{2} e^{-2t}$. Then, we have

$$\int_{0}^{+\infty} te^{-2t}\, dt = \frac{-1}{2} te^{-2t}|_{0}^{+\infty} + \frac{1}{2}\int_{0}^{+\infty} e^{-2t}\, dt$$

$$= 0 + \frac{-1}{4}[e^{-2t}|_{0}^{+\infty}$$

$$= \frac{-1}{4}[0 - 1] = \frac{1}{4}.$$

Now, definitions of some special functions arising from the computations will be given.

Definition 2.1.35. The formula given by

$$\Gamma(z) = \int_{0}^{+\infty} e^{-t}\, t^{z-1} dt, \qquad Re[z] > 0$$

defines the **Gamma** function $\Gamma(z)$ (Lebedev, 1972).

Definition 2.1.36. The formula given by

$$\mathbf{B}(x, y) = \int_{0}^{1} t^{x-1}\, (1 - t)^{y-1}\, dt$$

defines the **Beta** function $\mathbf{B}(x, y)$ for $Re[x] > 0$ and $Re[y] > 0$. In particular, the relationship between Gamma and Beta functions is given as

$$\mathbf{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}$$

for $Re[x] > 0$ and $Re[y] > 0$ (Lebedev, 1972).

Definition 2.1.37. Let B be a set such that $t \in B \subset (-\infty, \infty)$ and $h: B \to \mathbb{R}$ and $q: B \to \mathbb{R}$ with $q(t) \neq 0$ be two functions. Then, the following equation

$$\lim_{t \to t_0} \frac{h(t)}{q(t)} = 0$$

states that $h(t) = \boldsymbol{o}(q(t))$, where $t_0 \in [-\infty, +\infty]$. Here, the symbol "$\boldsymbol{o}$" is known in the literature as (Landau's) little o notation (Murray, 1984).

Example 2.1.24. Let $h, q: \mathbb{R}_+ \to \mathbb{R}$ be defined by $h(t) = t + 2021$ and $q(t) = t^2$. Since $\lim_{t \to +\infty} \frac{t+2021}{t^2} = 0$, $t + 2021 = \boldsymbol{o}(t^2)$, for sufficiently large positive real number t_1 such that $t > t_1$.

2.2. INTEGRAL OPERATORS AND AROUND

Definition 2.2.1. Let X and Y be two function spaces. A rule $\boldsymbol{L}$ associating each function g taken from X to a function h in Y is called an **operator** which is denoted by $\boldsymbol{L}(g; x) = h(x)$. Here, x is the element of the domain of h. The space X is called the domain of the operator $\boldsymbol{L}$ and is denoted by $X = \mathfrak{D}(\boldsymbol{L})$. The image set of $\boldsymbol{L}$ consisting of the functions h is denoted by $\mathcal{R}(\boldsymbol{L})$ being a subset of Y.

It is decided whether an operator is linear or not provided that the space X is a (linear) vector space as follows:

Let $g_1(x)$ and $g_2(x)$ be any two functions in $\mathfrak{D}(\boldsymbol{L})$, which are defined on a suitable domain compatible with $\mathfrak{D}(\boldsymbol{L})$, and a_1 and a_2 be arbitrary real or complex numbers. If the condition is given as:

$$\boldsymbol{L}(a_1 g_1 + a_2 g_2; x) = a_1 \boldsymbol{L}(g_1; x) + a_2 \boldsymbol{L}(g_2; x)$$

is fulfilled, then the operator $\boldsymbol{L}$ is called a **linear operator** (Hacısalihoğlu and Hacıyev, 1995).

Definition 2.2.2. Assume that (a, b) be an arbitrary interval in $\mathbb{R}$ whose endpoints satisfy the inequality $-\infty \leq a < b \leq +\infty$. Let g be a function defined on this interval. The expression given as

$$\boldsymbol{T}(g; p) = G(p) = \int_a^b g(t)\boldsymbol{K}(t, p)\, dt, \qquad (2.2.1)$$

where $t \in (a, b)$, defines the integral transform of g. $\boldsymbol{K}(t, p)$ is a kernel of the transform satisfying some properties for variables t and p. In order to construct the original function, the inverse of the transform given as $\boldsymbol{T}^{-1}\,(\boldsymbol{G}(p); t) = g(t)$ is used. Here, the process is done in the following way:

$$\boldsymbol{T}^{-1}\,(\boldsymbol{G}(p); t) = \boldsymbol{T}^{-1}\,(\boldsymbol{T}\,(g; p); t) = \boldsymbol{I}(g; t) = g(t),$$

where $\boldsymbol{I}$ is an identity transform (or operator) (Debnath and Bhatta, 2007).
Debnath and Bhatta (2007) showed the linearities of this unified form of integral transform and its inverse as follows:

Let $g_1(x)$ and $g_2(x)$ be any two functions in the domain of $\boldsymbol{T}$ and a_1 and a_2 be arbitrary real or complex numbers. Verification of linearity conditions gives that:

$$\begin{aligned}\boldsymbol{T}(a_1 g_1 + a_2 g_2; p) &= \int_a^b \big(a_1 g_1(t) + a_2 g_2(t)\big)\boldsymbol{K}(t, p)\, dt \\ &= a_1\boldsymbol{T}(g_1; p) + a_2\boldsymbol{T}(g_2; p) \\ &= a_1\boldsymbol{G_1}(p) + a_2\boldsymbol{G_2}(p),\end{aligned}$$

and in view of linearity of $\boldsymbol{T}$, one has

$$\begin{aligned}\boldsymbol{T}^{-1}\,(a_1\boldsymbol{G_1}(p) + a_2\boldsymbol{G_2}(p); t) &= \boldsymbol{T}^{-1}\,(a_1\boldsymbol{T}(g_1; p) + a_2\boldsymbol{T}(g_2; p); t) \\ &= \boldsymbol{T}^{-1}\,(\boldsymbol{T}(a_1 g_1; p) + \boldsymbol{T}(a_2 g_2; p); t) \\ &= \boldsymbol{T}^{-1}\,\big(\boldsymbol{T}(a_1 g_1 + a_2 g_2; p)\big) \\ &= \boldsymbol{I}(a_1 g_1 + a_2 g_2; t)\end{aligned}$$

$$= a_1 g_1(t) + a_2 g_2(t)$$
$$= a_1 \boldsymbol{T}^{-1}\left(\boldsymbol{G_1}(p); t\right) + a_2 \boldsymbol{T}^{-1}\left(\boldsymbol{G_2}(p); t\right).$$

Definition 2.2.3. Assume that (a, b) be an arbitrary interval in $\mathbb{R}$ whose endpoints satisfy the inequality $-\infty \leq a < b \leq +\infty$. Let g be a function defined on this interval. The expression given as

$$\mathbf{T}_{\boldsymbol{\sigma}}(g; y) = \int_a^b g(t)\mathbf{K}_\sigma(t, y)\, dt, \qquad (2.2.2)$$

where $\sigma \in \mathbb{N}$, $t \in [a, b]$ and $y \in (a, b)$, defines the linear singular integral. Here, the function $\mathbf{K}_{\boldsymbol{\sigma}}$ is a **kernel**, that is,

$$\lim_{\boldsymbol{\sigma}\to+\infty} \int_{a_1}^{b_1} \mathbf{K}_\sigma(t, y)\, dt = 1,$$

where $a \leq a_1 < y < b_1 \leq b$ (Natanson, 1960).

PART 3

FROM MELLIN TRANSFORMS TO MELLIN CONVOLUTION OPERATORS

In this part, theoretical information about Mellin transforms, and Mellin convolution operators will be given.

3.1. MELLIN AND FOURIER TRANSFORMS

Two definitions of Mellin transform of a function are as follows:

Definition 3.1.1. Let $f \in L^1_{Loc}(\mathbb{R}_+)$. The following formula

$$\mathcal{M}(f;s) = f_{\mathcal{M}}(s) = \int_0^{+\infty} f(t)t^{s-1}dt, \tag{3.1.1}$$

where $s = r + iv \in \mathbb{C}$ with $r, v \in \mathbb{R}$, is called the **Mellin transform** $f_{\mathcal{M}}$ of f (Flajolet et al., 1995).

Definition 3.1.2. The notation $\langle\alpha, \beta\rangle$ stands for open strip consisting of complex numbers $s = r + iv$ with $v \in \mathbb{R}$ and $\alpha < r < \beta$. In addition, the concept called **fundamental strip** stands for the largest open strip in which the Mellin transform $f_{\mathcal{M}}$ converges (Flajolet et al., 1995). In this thesis, we follow this notation. A similar strip concept is used in (Butzer and Jansche, 1997) with different symbolization.

Definition 3.1.3. The Mellin transform $f_{\mathcal{M}}$ of f, $f : \mathbb{R}_+ \to \mathbb{C}$ is defined by the following formula:

$$\mathcal{M}(f;s) = f_{\mathcal{M}}(s) = \int_{0}^{+\infty} f(t)t^{s-1} \ dt \tag{3.1.2}$$

provided that $f(t)t^{s-1} \in L^1\ (\mathbb{R}_+)$ for some $s = r + \ iv \ \in \mathbb{C}$ with $r, v \in \mathbb{R}$.
The **inverse Mellin transform** $\mathcal{M}_r^{-1}\ (g;t)$ of $g{:}\{r\} \times i\mathbb{R}_+ \to \mathbb{C}$ is given as:

$$\mathcal{M}_r^{-1}\ (g\ ;t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(r+iv)t^{-(r+iv)} \ dv, \tag{3.1.3}$$

where $g \in L^1\ (\{r\} \times i\mathbb{R}_+)$ (Butzer and Jansche, 1997).

Remark 3.1.1. The condition is given as "$f(t)t^{s-1} \in L^1\ (\mathbb{R}_+)$" is emphasized in Definition 3.1.3. It seems quite natural to consider the condition used in Definition 3.1.3 due to the characteristic properties of Lebesgue integrable functions. On the other hand, Butzer and Jansche (1997) simplified their condition as "$f(t)t^{r-1} \in L^1\ (\mathbb{R}_+)$" by proving the inequality given by $|\mathcal{M}\ (f;r+iv)| \ \leq \ \|f\|_{L^1_r(\mathbb{R}_+)}$, where $|.|$ denotes complex modulus, under the assumption $f\ (t)t^{r-1} \ \in L^1(\mathbb{R}_+)$, where $L^1_r(\mathbb{R}_+)$ is the function space consisting of the functions $f{:}\,\mathbb{R}_+ \to \mathbb{C}$ such that $f\ (t)t^{r-1} \in L^1(\mathbb{R}_+)$ for some $r \in \mathbb{R}$ with respect to the norm defined by $\|f\|_{L^1_r(\mathbb{R}_+)} = \int_0^{+\infty} |f(t)|t^{r-1}\, dt$ is finite for some $r \in \mathbb{R}$. In addition, by defining the space $L^1_r(\mathbb{R}_+)$ Butzer and Jansche (1997) gave a characterization for the class of functions whose Mellin transforms are convergent (resp. exist).

Now, some basic properties collected from the literature are given. Since Mellin and Fourier transforms have been widely studied over the years, and their properties have been given in various forms by many authors. The sources which we used to collect some of these features below can be found in Remarks 3.1.2 and 3.1.3.

Property 3.1.1. Recall that we included the unified forms of the integral transform and its inverse given by Debnath and Bhatta (2007). Since the same authors also proved the linearities of the indicated transforms, of this general case, it is stated in the same work that the Mellin transform, and its inverse are also linear operators (or transforms) (see (Debnath and Bhatta, 2007)). Now, replacing equation (2.2.1) by Mellin transform formula, we give the particular case of their linearity proof as follows:

Let f_1 and f_2 be any two functions in $L_r^1(\mathbb{R}_+)$ for some r and a_1 and a_2 be arbitrary real or complex numbers. Since

$$\mathcal{M}(f;s) = \int_0^{+\infty} f(t)t^{s-1}dt,$$

where $s = r + iv \in \mathbb{C}$ with $r, v \in \mathbb{R}$, we easily verify the linearity conditions in the following style:

$$\begin{aligned}\mathcal{M}(a_1f_1 + a_2f_2;s) &= \int_0^{+\infty} (a_1f_1(t) + a_2f_2(t))t^{s-1}\ dt \\ &= a_1\int_0^{+\infty} f_1(t)\ t^{s-1}\ dt + a_2\int_0^{+\infty} f_2(t)\ t^{s-1}\ dt \\ &= a_1\mathcal{M}(f_1;s)\ +\ a_2\mathcal{M}(f_2;s).\end{aligned}$$

This property is known as the **linearity property** of the Mellin transform.

Property 3.1.2. $\mathcal{M}(f(bt);s) = b^{-s}f_{\mathcal{M}}(s)$, where $b \in \mathbb{R}_+$, $s \in \langle\alpha,\beta\rangle$ and $f \in L_r^1(\mathbb{R}_+)$. This property is known as the scaling property of Mellin transform.
Applying definition and variable change $bt = u$ indeed gives that:

$$\mathcal{M}(f(bt);s) = \int_0^{+\infty} f(bt)t^{s-1}dt = \int_0^{+\infty} f(u)u^{s-1}b^{1-s}b^{-1}du = b^{-s}f_{\mathcal{M}}(s).$$

Property 3.1.3. $\mathcal{M}(t^bf(t);s) = f_{\mathcal{M}}(s+b)$, where $b \in \mathbb{C}$, $s \in \langle\alpha - Re[b], \beta - Re[b]\rangle$ and $f \in L_r^1(\mathbb{R}_+)$.

Applying definition indeed gives that

$$\mathcal{M}(t^b f(t);s)=\int_0^{+\infty} t^b f(t) t^{s-1}dt=\int_0^{+\infty} f(t) t^{s+b-1}dt=f_{\mathcal{M}}(s+b).$$

Property 3.1.4. $\mathcal{M}(f(t^b);s)=b^{-1}f_{\mathcal{M}}(sb^{-1})$, where $b\in\mathbb{R}_+$, $s\in\langle b\alpha, b\beta\rangle$ and $f\in L_r^1(\mathbb{R}_+)$.

Applying definition and variable change $t^b=u$ indeed gives that:

$$\mathcal{M}(f(t^b);s)=\int_0^{+\infty} f(t^b) t^{s-1}dt=b^{-1}\int_0^{+\infty} f(u) u^{sb^{-1}-1}du=b^{-1}f_{\mathcal{M}}(sb^{-1}).$$

Property 3.1.5. $\mathcal{M}(t^{-1}f(t^{-1});s)=f_{\mathcal{M}}(1-s)$, where $f(t)t^{-s}\in L^1(\mathbb{R}_+)$ for some suitable complex number(s) s.

Applying definition and using the variable change $t^{-1}=u$ indeed gives that:

$$\mathcal{M}(t^{-1}f(t^{-1});s)=\int_0^{+\infty} t^{-1}f(t^{-1}) t^{s-1}dt=\int_0^{+\infty} f(u) u^{(1-s)-1}du=f_{\mathcal{M}}(1-s).$$

Property 3.1.6. Let $n\in\mathbb{N}$ be fixed and $f^{(n)}\equiv\frac{d^n}{dt^n}f(t)$ with $f^{(0)}\equiv f$. Assume that $\lim_{t\to+\infty} t^{s-i}f^{(n-i)}(t)=0$ and $\lim_{t\to 0^+} t^{s-i}f^{(n-i)}(t)=0$ for $i=1,2,\dots n$.
Then, there holds

$$\mathcal{M}\big(f^{(n)}(t);s\big)=(-1)^n\frac{\Gamma(s)}{\Gamma(s-n)}f_{\mathcal{M}}(s-n)$$

under the required conditions of existence of Mellin transform.

To see the relation holds true, we simply utilize mathematical induction. Let $n = 1$. By definition, we write

$$\mathcal{M}\left(f^{(1)}(t); s\right) = \int_0^{+\infty} t^{s-1} f^{(1)}(t)\, dt.$$

Integrating by parts gives:

$$\begin{aligned}\int_0^{+\infty} t^{s-1} f^{(1)}(t)\, dt &= [t^{s-1} f(t)]_0^{+\infty} - (s-1)\int_0^{+\infty} t^{s-2}\, f(t) dt \\ &= 0 - (s-1)\int_0^{+\infty} t^{(s-1)-1}\, f(t) dt \\ &= -(s-1) f_{\mathcal{M}}(s-1).\end{aligned}$$

Let $n = 2$. By the result for $n = 1$, we have

$$\begin{aligned}\mathcal{M}\left(f^{(2)}(t); s\right) &= -(s-1) f_{\mathcal{M}}^{(1)}(s-1). \\ &= (s-1)(s-2)\, f_{\mathcal{M}}(s-2).\end{aligned}$$

Suppose that the result holds true for $n = k - 1$. We will show that it is true for $n = k$. We have

$$\begin{aligned}\mathcal{M}\left(f^{(k-1)}(t); s\right) &= (-1)^{k-1}(s-1)(s-2) \ldots (s-k+1)\, f_{\mathcal{M}}(s-k+1) \\ &= (-1)^{k-1} \frac{\Gamma(s)}{\Gamma(s-k+1)} f_{\mathcal{M}}(s-k+1).\end{aligned}$$

Thus, we easily get the result, that is,

$$\begin{aligned}\mathcal{M}\left(f^{(k)}(t); s\right) &= -(s-1) f_{\mathcal{M}}^{(k-1)}(s-1). \\ &= -(s-1)(-1)^{k-1}(s-2)(s-3) \ldots (s-k)\, f_{\mathcal{M}}(s-k) \\ &= (-1)^k \frac{\Gamma(s)}{\Gamma(s-k)} f_{\mathcal{M}}(s-k).\end{aligned}$$

Property 3.1.7. Let $n \in \mathbb{N}$ and ${f_{\mathcal{M}}}^{(n)}(s) \equiv \frac{d^n}{ds^n} f_{\mathcal{M}}$ with ${f_{\mathcal{M}}}^{(0)} \equiv f_{\mathcal{M}}$. Then, $\mathcal{M}((\ln(\mathrm{t}))^n f(t); s) = (f_{\mathcal{M}}(s))^{(n)}$ under the required conditions of existence of Mellin transform.

Using the clue given by $\frac{d}{ds}(t^{s-1}) = \ln(t) t^{s-1}$ of (Debnath and Bhatta, 2007), we observe that $\frac{d^n}{ds^n}(t^{s-1}) = (\ln(t))^n t^{s-1}$. By definition, the claim holds as such:

$$\int_0^{+\infty} t^{s-1}(\ln(\mathrm{t}))^n f(t)\, dt = \frac{d^n}{ds^n}\int_0^{+\infty} t^{s-1} f(t)\, dt = (f_{\mathcal{M}}(s))^{(n)}.$$

Property 3.1.8. Under the assumptions of Property 3.1.3 and Property 3.1.6, there holds:

$$\mathcal{M}\left(t^n f^{(n)}(t); s\right) = (-1)^n s^n f_{\mathcal{M}}(s).$$

To see this, we simply utilize mathematical induction. Let $n = 1$. By Property 3.1.3, we write

$$\mathcal{M}\left(t f^{(1)}(t); s\right) = \mathcal{M}\left(f^{(1)}(t); s + 1\right).$$

Besides, since $\mathcal{M}\left(f^{(1)}(t); s\right) = -(s-1) f_{\mathcal{M}}(s-1)$ by Property 3.1.6, replacing here s by $s + 1$, we get:

$$\mathcal{M}\left(f^{(1)}(t); s + 1\right) = -s\, f_{\mathcal{M}}(s).$$

Therefore, we get:

$$\mathcal{M}\left(t f^{(1)}(t); s\right) = -s\, f_{\mathcal{M}}(s).$$

Let $n = 2$. By the result for $n = 1$, we have

$$\mathcal{M}\left(t^2 f^{(2)}(t); s\right) = -s\left(-s\, f_{\mathcal{M}}(s)\right)$$
$$= s^2\, f_{\mathcal{M}}(s).$$

Suppose that the result holds true for $n = k - 1$. We will show that it is true for $n = k$. We have

$$\mathcal{M}\left(t^{k-1} f^{(k-1)}(t); s\right) = (-1)^{k-1}\, s^{k-1}\, f_{\mathcal{M}}(s).$$

For $n = k$, we obtain the desired result, that is,

$$\mathcal{M}\left(t^k f^{(k)}(t); s\right) = -s\mathcal{M}\left(t^{k-1} f^{(k-1)}(t); s\right)$$
$$= -s(-1)^{k-1}\, s^{k-1}\, f_{\mathcal{M}}(s)$$
$$= (-1)^k\, s^k\, f_{\mathcal{M}}(s).$$

Remark 3.1.2. The Properties 3.1.1-3.1.4 given above can be found in Proposition 1 (Butzer and Jansche, 1997, pp. 336-337). The Properties 3.1.1-3.1.7 can be found in Theorem 1.4 of (Mamedov, 1991, p. 16) and (Mamedov, 1991, p. 60). The Properties 3.1.1-3.1.8 can also be found with most of their proofs in (Debnath and Bhatta, 2007, pp. 343-345). The written proofs (or clarifications) for properties discussed above, which are directly based on definitions of the concepts, can be seen and compared in the cited sources in this remark.

Property 3.1.9. $\mathcal{M}(f(t); s) = \Gamma(s)\, \lambda(-s)$, where

$$f(t) = \sum_{m=0}^{+\infty} \frac{\lambda(m)}{m!} (-1)^m (t)^m = \frac{\lambda(0)}{1} - \frac{\lambda(1)}{1!} t + \frac{\lambda(2)}{m!} t^2 - \cdots$$

and $\lambda(m)$ is a function associated to the series representation (see (Hardy, 1940; Amdeberhan et al., 2012)).

Remark 3.1.3. Property 3.1.9, known in the literature as "Ramanujan's Master Theorem", named after famous mathematician Srinivasa Ramanujan and its existence hypotheses were widely discussed in (Amdeberhan et al., 2012); see also (Hardy, 1940).

Example 3.1.1. By taking $f(t) = (1+t)^{-a}$ with $a > 0$ and $t \in \mathbb{R}_+$, Amdeberhan et al. (2012) showed that $\int_0^{+\infty} \frac{t^{s-1}}{(1+t)^a}\, dt = \Gamma(s)\lambda(-s)$, where $\lambda(-s) = \frac{\Gamma(a-s)}{\Gamma(a)}$ by using Binomial theorem including negative exponents:

$$(1+t)^{-a} = \sum_{m=0}^{+\infty} \binom{m+a-1}{m} (-1)^m t^m = \sum_{m=0}^{+\infty} \frac{\Gamma(m+a)}{\Gamma(a)} \frac{(-1)^m t^m}{m!},$$

where $\lambda(m) = \frac{\Gamma(m+a)}{\Gamma(a)}$ and replacing m by $-s$ in accordance with Ramanujan's Master Theorem.

Direct computation of the Mellin transform indeed gives (see, for example, (Brychkov et al., 2019)):

$$\int_0^{+\infty} \frac{t^{s-1}}{(1+t)^a}\, dt = \mathbf{B}(s, a-s) = \frac{\Gamma(s)\,\Gamma(a-s)}{\Gamma(a)}$$

with $a > Re[s]$ and $Re[s] > 0$.

Following the same steps, one obtains by taking $f(t) = (b+ct)^{-a}$ with $a, b, c > 0$ and $t \in \mathbb{R}_+$, $\int_0^{+\infty} \frac{t^{s-1}}{(b+ct)^a}\, dt = \Gamma(s)\lambda(-s)$, where $\lambda(-s) = b^{-(a-s)} c^{-s} \frac{\Gamma(a-s)}{\Gamma(a)}$ by using well-known Binomial series of the function:

$$\begin{aligned}(b+ct)^{-a} &= \sum_{m=0}^{+\infty} \binom{m+a-1}{m} b^{-(m+a)} c^m (-1)^m t^m \\ &= \sum_{m=0}^{+\infty} \frac{\Gamma(m+a)}{\Gamma(a)} \frac{c^m b^{-(m+a)} (-1)^m t^m}{m!},\end{aligned}$$

where

$$\lambda(m) = \frac{\frac{b^{-(m+a)}}{c^{-m}}\Gamma(m+a)}{\Gamma(a)}.$$

Direct computation of the Mellin transform indeed gives (see, for example, (Brychkov et al., 2019)):

$$\int_{0}^{+\infty} \frac{t^{s-1}}{(b+ct)^{a}}\,dt = \frac{b^{-(a-s)}c^{-s}\Gamma(\mathrm{s})\,\Gamma(a-s)}{\Gamma(a)} = b^{-(a-s)}c^{-s}\mathbf{B}(\mathrm{s}, a-s)$$

with $a > Re[s]$ and $Re[s] > 0$.

Here, derivation of Binomial series of $f(t) = (1+t)^{-a}$ with $a > 0$ and for some $t \in \mathbb{R}_{+}$ is as follows:

Using well-known Taylor formula whose general term is $\frac{f^{(m)}(x_0)}{m!}(t-x_0)^m$ with $x_0 = 0$, the series (i.e., Maclaurin series) will seem as

$$(1+t)^{-a} = \sum_{m=0}^{+\infty} f^{(m)}(0)\frac{t^m}{m!}.$$

Using the Taylor formula, we obtain $\frac{f^{(m)}(0)}{m!} = \binom{-a}{m}$ for $m = 0, 1, \ldots$.

Using identity $\binom{-a}{m} = \binom{m+a-1}{m}(-1)^m$ for integers $m \geq 0$ (see, for example, (Wolfram Research, 1988)), we obtain the desired result, that is,

$$(1+t)^{-a} = \sum_{m=0}^{+\infty} \binom{-a}{m} t^m = \sum_{m=0}^{+\infty} \binom{m+a-1}{m} (-1)^m t^m,$$

where $0 < t < 1$ in view of the ratio test.

The definition of Mellin derivative given by Mamedov (1991) is as follows:

Definition 3.1.4. The Mellin derivative of a function g defined on a suitable subset of $\mathbb{R}_+$ at a point $t \in \mathbb{R}_+$ belonging to its domain is defined by

$$\mathbf{M}[g(t)] = (-t)\frac{d}{dt}g(t),$$

and in the limit form:

$$\mathbf{M}[g(t)] = \lim_{k\to 1}\frac{g\left(\frac{t}{k}\right) - g(t)}{\ln(k)}$$

provided that this limit exists at a point $t \in \mathbb{R}_+$ (Mamedov, 1991).
In fact, making the variable change and using well-known L'Hospital's rule in the limit, one has

$$\begin{aligned}\mathbf{M}[g(t)] &= \lim_{k\to 1}\frac{g\left(\frac{t}{k}\right) - g(t)}{\ln(k)} \\ &= \lim_{k\to 1}\frac{\left(\frac{t}{k} - t\right)}{\ln(k)}\lim_{k\to 1}\frac{g\left(\frac{t}{k}\right) - g(t)}{\left(\frac{t}{k} - t\right)} \\ &= (-t)\lim_{u\to t}\frac{g(u) - g(t)}{(u - t)} = (-t)\frac{d}{dt}g(t).\end{aligned}$$

For example, $\mathbf{M}[e^t] = -te^t$ at each point $t \in \mathbb{R}_+$.
Another definition of Mellin derivative with its detailed treatment can be found in (Butzer and Jansche, 1997).

Example 3.1.2. Let $f: \mathbb{R}_+ \to \mathbb{R}$ be given by

$$f(t) = \begin{cases} t, & t \in (0,1], \\ 0, & t \in \mathbb{R}_+ \setminus (0,1]. \end{cases}$$

Since the integral evaluated as

$$\int_0^{+\infty} |f(t)|t^{r-1}dt = \int_0^1 tt^{r-1}dt$$
$$= \frac{1}{1+r}.$$

is absolutely convergent for $r > -1$, the Mellin transform $f_{\mathcal{M}}$ of f exists for these values. That is, for $Re[s] > -1$, there holds:

$$f_{\mathcal{M}}(s) = \int_0^{+\infty} f(t)t^{s-1}dt$$
$$= \int_0^1 tt^{s-1}dt$$
$$= \frac{1}{1+s} .$$

Example 3.1.3. Let $f_m\,(t) = t^m e^{-t}$ with $t \in \mathbb{R}_+$ and $m = 0, 1$. Let $m = 0$. The definition of well-known Gamma function $\Gamma(s)$ is given by $f_{\mathcal{M}}$, that is,

$$\mathcal{M}(f_0; s) = \int_0^{+\infty} f_0(t)t^{s-1}dt$$
$$= \int_0^{+\infty} e^{-t}t^{s-1}dt$$
$$= \Gamma(s).$$

This integral converges absolutely for $Re[s] > 0$, that is, here $r \in \langle 0, +\infty\rangle$. Now, let $m = 1$,

$$\mathcal{M}(f_1; s) = \int_0^{+\infty} f_1(t)t^{s-1}dt$$

$$= \int_{0}^{+\infty} e^{-t} t^{s} dt$$

$$= \Gamma(s+1).$$

The recursion formula written as $\Gamma(s+1) = s\Gamma(s)$ is obtained using integration by parts several times. Clearly, using the inversion formula there holds that:

$$e^{-t} = \frac{1}{2i\pi} \int_{r-i\infty}^{r+i\infty} \Gamma(\mathrm{s}) t^{-s}\, ds$$

for $r > 0$; see (Lebedev, 1972; Flajolet et al., 1995; Butzer and Jansche, 1997; Debnath and Bhatta, 2007).

Example 3.1.4. Let $f: \mathbb{R}_{+} \to \mathbb{R}$ be given by $f(t) = e^{-t^2}$. Its Mellin transform is computed as follows:

$$f_{\mathcal{M}}(s) = \int_{0}^{+\infty} f(t) t^{s-1} dt$$

$$= \int_{0}^{+\infty} e^{-t^2} t^{s-1} dt\,.$$

Making the variable change $t^s = v$ with $st^{s-1}dt = dv$ and $Re[s] > 0$, we have

$$\int_{0}^{+\infty} e^{-\left(v^{\frac{1}{s}}\right)^2} \frac{1}{s} dv = \frac{1}{s} \int_{0}^{+\infty} e^{-v^{\frac{2}{s}}} dv\,.$$

One more changing of variables $v^{\frac{2}{s}} = u$ with $\frac{2}{s} v^{\frac{2}{s}-1} dv = du$ and $dv = \frac{s}{2} u^{\frac{s}{2}-1}\, du$ gives:

$$\frac{1}{s}\int_{0}^{+\infty} e^{-v^{\frac{2}{s}}}\,dv = \frac{1}{2}\int_{0}^{+\infty} e^{-u}u^{\frac{s}{2}-1}\,du = \frac{1}{2}\Gamma\left(\frac{s}{2}\right), \qquad Re[s] > 0.$$

In order to give the **relationship** between Mellin and Fourier transforms, the following definition is necessarily given.

Definition 3.1.5. Let $f \in L^1(\mathbb{R})$ be a given function. The **Fourier transform** $\mathcal{F}(f;k)$ is defined by:

$$\mathcal{F}(f;k) = f_{\mathcal{F}}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ikx} f(x)dx,$$

where $k \in \mathbb{R}$. The formula given as:

$$\mathcal{F}^{-1}\big(f_{\mathcal{F}}(k)\big) = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ikx} f_{\mathcal{F}}(k)\,dk$$

is the **inverse Fourier transform** (see, for example, (Debnath and Bhatta, 2007; Altın, 2011)).

The required conditions under which the above representations are convergent also discussed in (Debnath and Bhatta, 2007; Altın, 2011).

As in Property 3.1.1, we show that the Fourier transform is linear. Let f_1 and f_2 be any two functions in $L^1(\mathbb{R})$ and a_1 and a_2 be arbitrary real or complex numbers. Since

$$\mathcal{F}(f;k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ikx} f(x)dx,$$

we simply have

$$\mathcal{F}(a_1 f_1 + a_2 f_2; k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} (a_1 f_1(x) + a_2 f_2(x)) dx$$

$$= \frac{a_1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} \ f_1(x) \ dx + \frac{a_2}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} \ f_2(x) \ dx$$

$$= a_1 \mathcal{F}(f_1; k) + a_2 \mathcal{F}(f_2; k).$$

Example 3.1.5. Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

The Fourier transform of f is computed in the following way:

$$f_{\mathcal{F}}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx} f(x) dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\int_{-\infty}^{-1} e^{-ikx} f(x) dx \right) + \left(\int_{-1}^{+1} e^{-ikx} f(x) dx \right) + \left(\int_{+1}^{+\infty} e^{-ikx} f(x) dx \right) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\int_{-\infty}^{-1} e^{-ikx} (0) dx \right) + \left(\int_{-1}^{+1} e^{-ikx} (1) dx \right) + \left(\int_{+1}^{+\infty} e^{-ikx} (0) dx \right) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 + \left(\int_{-1}^{+1} e^{-ikx} dx \right) + 0 \right]$$

$$= (\frac{-1}{\sqrt{2\pi} \ ik}) \left[e^{-ikx} \Big|_{-1}^{1} \right.$$

$$= \left(\frac{-1}{\sqrt{2\pi} \ ik} \right) [e^{-ik} - e^{ik}]$$

$$= \left(\frac{-1}{\sqrt{2\pi} \ ik} \right) [(\cos k - i \sin k) - (\cos k + i \sin k)]$$

$$= \left(\frac{-1}{\sqrt{2\pi} \ ik} \right) [\cos k - i \sin k - \cos k - i \sin k]$$

$$= \left(\frac{-1}{\sqrt{2\pi} \ ik} \right) [-2i \sin k]$$

$$= \frac{2\sin k}{\sqrt{2\pi}\ k} = \sqrt{\frac{2}{\pi}}\ \frac{\sin k}{k}.$$

For a more general version of this example, see (Altin, 2011).

Example 3.1.6. Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^{-|x|}$, since

$$\|f\|_{L^1(\mathbb{R})} = \int_{-\infty}^{+\infty} e^{-|x|}\, dx$$
$$= 2 < +\infty\,,$$

we have $f \in L^1(\mathbb{R})$.

Its Fourier transform is computed as follows:

$$f_{\mathcal{F}}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ikx}\, e^{-|x|}\, dx$$
$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{-ikx}\, e^{-(-x)}\, dx + \int_{0}^{+\infty} e^{-ikx}\, e^{-(x)}\, dx \right]$$
$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{x-ikx}\ dx + \int_{0}^{+\infty} e^{-x-ikx}\ dx \right]$$
$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{(1-ik)x}\, dx + \int_{0}^{+\infty} e^{-(1+ik)x}\, dx \right]$$
$$= \frac{1}{\sqrt{2\pi}} \left[[\frac{1}{1-ik} e^{(1-ik)x}\Big|_{-\infty}^{0} + [\frac{-1}{1+ik} e^{-(1+ik)x}\Big|_{0}^{+\infty} \right]$$
$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{1-ik}\ (1-0) - \frac{1}{1+ik}\ (0-1) \right]$$
$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{1-ik} + \frac{1}{1+ik} \right]$$
$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{1+k^2} \right), \quad k \in \mathbb{R}.$$

On the other hand, using the inverse Fourier transform, one has

$$e^{-|x|} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}\frac{2}{1+k^2}\,dk\,,\;\; x\in\mathbb{R}\,.$$

For a more general version of this example and some other well-known functions, Fourier transforms (see, for example, (Debnath and Bhatta, 2007)).

Debnath and Bhatta (2007) expressed the relationship between the Mellin and Fourier transforms in the following fashion:

Changing the integral variables as $e^x = t$ with $dx = \frac{dt}{t}$ and $ik = r - s$ in the expression defined in Fourier transform, one has

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ikx} f(x)dx = \frac{1}{\sqrt{2\pi}}\int_{0}^{+\infty} t^{-(r-s)}\frac{1}{t}f(\ln t)dt$$
$$= f_{\mathcal{F}}(is - ir).$$

Replacing $\frac{t^{-r}}{\sqrt{2\pi}}f(\ln t)$ by $g(t)$ and $f_{\mathcal{F}}(is - ir)$ by $g_{\mathcal{M}}(s)$, one obtains the definition of Mellin transform, that is,

$$g_{\mathcal{M}}(s) = \int_{0}^{+\infty} g(t)t^{s}\frac{dt}{t}\,.$$

In view of similar considerations, the relationship between the inverse Mellin and inverse Fourier transforms can be obtained. This relationship and the relationships between Laplace, bilateral Laplace and Mellin transforms were examined in (Butzer and Jansche, 1997).

3.2. MELLIN-TYPE CONVOLUTIONS

Now, some basic definitions are given.

Definition 3.2.1. Let $h, g: \mathbb{R}_+ \to \mathbb{C}$ be two functions. The functional product expressed as $h *_M g$ defined by

$$(h *_M g)(t) = \int_0^{+\infty} h\left(\frac{t}{\omega}\right) g(\omega) \frac{d\omega}{\omega},$$

where $t \in \mathbb{R}_+$ is called the **Mellin convolution product** provided that the integral exists (Mamedov, 1991; Butzer and Jansche, 1997).

Definition 3.2.2. The functional product expressed as $h * g$ defined by

$$(h * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h(x - t)\, g(t) dt,$$

where $x \in \mathbb{R}$, is called **the convolution** of the functions h and g defined on $\mathbb{R}$ whenever the integral exists (Altın, 2011).

These are classical convolutions. However, we want to mention here one more convolution in the sense of Mellin.

Definition 3.2.3. Let $h, g: \mathbb{R}_+ \to \mathbb{C}$ be two functions. The operation defined by

$$(h \odot g)(t) = \int_0^{+\infty} h(\omega t) g(\omega)\, d\omega,$$

where $t \in \mathbb{R}_+$ is another Mellin-type convolution product (Debnath and Bhatta, 2007).

Classical Mellin and usual convolutions have been widely studied over the years, and their properties have been given in various forms by many authors. The sources which we used to collect some of these features below can be found in Remark 3.2.1.

Now, some basic properties collected from the literature are given.

Property 3.2.1. The classical Mellin convolution product $*_M$ is commutative.
Let $h, g: \mathbb{R}_+ \to \mathbb{C}$ be two functions. In fact, making the variable change $\omega = \frac{t}{z}$ with $d\omega = -\frac{tdz}{z^2}$, where $z, t > 0$, we have

$$(h *_M g)(t) = \int_0^{+\infty} h\left(\frac{t}{\omega}\right) g(\omega) \frac{d\omega}{\omega} = \int_0^{+\infty} g\left(\frac{t}{z}\right) h(z) \frac{dz}{z} = (g *_M h)(t).$$

Property 3.2.2. The classical convolution product $*$ is commutative.
Let h, g be two functions defined on $\mathbb{R}$. In fact, making the variable change $x - t = w$ with $-dt = dw$, where $w, x \in \mathbb{R}$, we have

$$\begin{aligned}(h * g)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h(x - t) g(t)\, dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x - w) h(w)\, dw = (g * h)(x).\end{aligned}$$

Property 3.2.3. Assume that $h, g \in L_r^1(\mathbb{R}_+)$ for some $r \in \mathbb{R}$. Then, for classical Mellin convolution product $*_M$, there holds:

$(h *_M g)_{\mathcal{M}}(s) = h_{\mathcal{M}}(s) g_{\mathcal{M}}(s),$
where $s = r + iv \in \mathbb{C}$ with $r, v \in \mathbb{R}$.

Let $h, g: \mathbb{R}_+ \to \mathbb{C}$ be two functions. Consider the functions for $\omega > 0$

$$h^1\left(\frac{t}{\omega}\right) = \begin{cases} \left(\frac{t}{\omega}\right)^{s-1} h\left(\frac{t}{\omega}\right), & if \quad t \in \mathbb{R}_+, \\ 0, & if\ t \in \mathbb{R} \setminus \mathbb{R}_+, \end{cases}$$

and

$$g^1(\omega) = \begin{cases} \omega^{-1} g(\omega), & if\ \omega \in \mathbb{R}_+, \\ 0, & if\ \omega \in \mathbb{R} \setminus \mathbb{R}_+. \end{cases}$$

Since

$$(h *_M g)_{\mathcal{M}}(s) = \int_0^{+\infty} t^{s-1} \left(\int_0^{+\infty} h\left(\frac{t}{\omega}\right) g(\omega) \frac{d\omega}{\omega} \right) dt$$

$$= \omega^{s-1} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} h^1\left(\frac{t}{\omega}\right) g^1(\omega) d\omega \right) dt,$$

in view of Fubini's theorem (Theorem 2.1.2), we may write

$$\omega^{s-1} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} h^1\left(\frac{t}{\omega}\right) g^1(\omega) d\omega \right) dt = \omega^{s-1} \left(\int_{-\infty}^{+\infty} g^1(\omega) d\omega \right) \left(\int_{-\infty}^{+\infty} h^1\left(\frac{t}{\omega}\right) dt \right)$$

$$= \left(\int_0^{+\infty} g(\omega) \frac{d\omega}{\omega} \right) \left(\int_0^{+\infty} t^{s-1}\ h\left(\frac{t}{\omega}\right) dt \right).$$

In fact, making a change of variable $z = \frac{t}{\omega}$ with $dz = \frac{dt}{\omega}$, where $z > 0$, in the second integral above, we have

$$\left(\int_0^{+\infty} g(\omega) \frac{d\omega}{\omega} \right) \left(\int_0^{+\infty} t^{s-1}\ h\left(\frac{t}{\omega}\right) dt \right)$$

$$= \left(\int_0^{+\infty} g(\omega) \omega^{s-1}\, d\omega \right) \left(\int_0^{+\infty} h(z) z^{s-1}\, dz \right)$$

$$= g_{\mathcal{M}}(s)\ h_{\mathcal{M}}(s) = h_{\mathcal{M}}(s) g_{\mathcal{M}}(s) = (h *_M g)_{\mathcal{M}}(s).$$

Property 3.2.4. For the classical convolution product $*$, there holds

$(h * g)_{\mathcal{F}}(k) = h_{\mathcal{F}}(k) g_{\mathcal{F}}(k),$

where $k \in \mathbb{R}$, provided that Fourier transform exists.

We will prove that by using the definition. Since

$$(h * g)_{\mathcal{F}}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h(x-t)\, g(t) dt \right) e^{-ikx} dx,$$

in view of Fubini's theorem (Theorem 2.1.2), we may write

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h(x-t)\, g(t) dt \right) e^{-ikx} dx$$

$$= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(t) dt \right) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h(x-t)\, e^{-ikx} dx \right).$$

In fact, making the variable change as $w = x - t$ with $x = w + t$ and $dx = dw$ in the second integral above, we have:

$$\begin{aligned}
(h * g)_{\mathcal{F}}(k) &= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(t) dt \right) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h(w)\, e^{-ik(w+t)} dw \right) \\
&= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(t)\, e^{-ikt} dt \right) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h(w) e^{-ikw}\, dw \right) \\
&= g_{\mathcal{F}}(k) h_{\mathcal{F}}(k) = h_{\mathcal{F}}(k) g_{\mathcal{F}}(k) = (g * h)_{\mathcal{F}}(k).
\end{aligned}$$

Property 3.2.5. The Mellin-type convolution $\circledcirc$ has the following property:

$$(h \circledcirc g)_{\mathcal{M}}(s) = h_{\mathcal{M}}(s)\, g_{\mathcal{M}}(1-s),$$

where $s = r + iv \in \mathbb{C}$ with $r, v \in \mathbb{R}$, provided that the Mellin transform exists.

Using Fubini's theorem (Theorem 2.1.2), similar to that of Property 3.2.3, we have

$$(h \circledcirc g)_{\mathcal{M}}(s) = \int_0^{+\infty} t^{s-1} \left(\int_0^{+\infty} h(\omega t)\, g(\omega)\, d\omega \right) dt$$
$$= \left(\int_0^{+\infty} g(\omega) d\omega \right) \left(\int_0^{+\infty} t^{s-1}\, h(\omega t) dt \right).$$

Now, making the variable change $\varphi = \omega t$ and $d\varphi = \omega dt$ with $\omega > 0$ in the second integral above, the result follows, that is, $(f \circledcirc g)_{\mathcal{M}}(s) = f_{\mathcal{M}}(s) g_{\mathcal{M}}(1-s)$.

Property 3.2.6. Let $h, g \in \mathcal{L}^1(\mathbb{R}_+)$ and $f \in \mathcal{L}^p(\mathbb{R}_+)$ with $1 \leq p < +\infty$. Then, one has:

$$\|h *_M g\|_{\mathcal{L}^1(\mathbb{R}_+)} \leq \|h\|_{\mathcal{L}^1(\mathbb{R}_+)} \|g\|_{\mathcal{L}^1(\mathbb{R}_+)}$$

and

$$\|f *_M g\|_{\mathcal{L}^p(\mathbb{R}_+)} \leq \|f\|_{\mathcal{L}^p(\mathbb{R}_+)} \|g\|_{\mathcal{L}^1(\mathbb{R}_+)}.$$

Using Fubini's theorem (Theorem 2.1.2), similar to that of Property 3.2.3 and the definition of norm, we have

$$\|h *_M g\|_{\mathcal{L}^1(\mathbb{R}_+)} = \int_0^{+\infty} \left| \int_0^{+\infty} h\left(\frac{t}{\omega}\right) g(\omega) \frac{d\omega}{\omega} \right| \frac{dt}{t}$$
$$\leq \left(\int_0^{+\infty} \left| h\left(\frac{t}{\omega}\right) \right| \frac{dt}{t} \right) \left(\int_0^{+\infty} |g(\omega)| \frac{d\omega}{\omega} \right).$$

Now, making the variable change $\omega\varphi = t$ and $\omega d\varphi = dt$ with $\varphi > 0$ in the second integral above, the result follows, that is,

$$\|h *_M g\|_{\mathcal{L}^1(\mathbb{R}_+)} \leq \|h\|_{\mathcal{L}^1(\mathbb{R}_+)} \|g\|_{\mathcal{L}^1(\mathbb{R}_+)}.$$

For the second one, using Hölder-Minkowski inequality (Theorem 2.1.3) and definition of norm, we have

$$\|f *_M g\|_{\mathcal{L}^p(\mathbb{R}_+)} = \left(\int_0^{+\infty} \left| \int_0^{+\infty} f\left(\frac{t}{\omega}\right) g(\omega) \frac{d\omega}{\omega} \right|^p \frac{dt}{t} \right)^{1/p}$$
$$\leq \left(\int_0^{+\infty} |g(\omega)| \frac{d\omega}{\omega} \right) \left(\int_0^{+\infty} \left| f\left(\frac{t}{\omega}\right) \right|^p \frac{dt}{t} \right)^{1/p}.$$

Now, making the variable change $\omega\varphi = t$ and $\omega d\varphi = dt$ with $\varphi > 0$ in the second integral above, the result follows, that is,

$$\|f *_M g\|_{\mathcal{L}^p(\mathbb{R}_+)} \leq \|f\|_{\mathcal{L}^p(\mathbb{R}_+)} \|g\|_{\mathcal{L}^1(\mathbb{R}_+)}.$$

Property 3.2.7. Let $h, g \in L^1(\mathbb{R})$ and $f \in L^p(\mathbb{R})$ with $1 \leq p < +\infty$. Then, one has $\|h * g\|_{L^1(\mathbb{R})} \leq \|h\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}$

and

$$\|f * g\|_{L^p(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}.$$

Using Fubini's theorem (Theorem 2.1.2) and definition of norm, we have

$$\|h * g\|_{L^1(\mathbb{R})} = \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} h(t-\omega)\, g(\omega) d\omega \right| dt$$
$$\leq \left(\int_{-\infty}^{+\infty} |g(\omega)|\, d\omega \right) \left(\int_{-\infty}^{+\infty} |h(t-\omega)|\, dt \right).$$

Now, making the variable change $\omega + \varphi = t$ and $d\varphi = dt$ with $\varphi \in \mathbb{R}$ in the second integral above, the result follows, that is,

$$\|h * g\|_{L^1(\mathbb{R})} \le \|h\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}.$$

For the second one, using Hölder-Minkowski inequality (Theorem 2.1.3) and definition of norm, we have

$$\begin{aligned} \|f * g\|_{L^p(\mathbb{R})} &= \left(\int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} f(t-\omega)\, g(\omega) d\omega \right|^p dt \right)^{1/p} \\ &\le \left(\int_{-\infty}^{+\infty} |g(\omega)|\, d\omega \right) \left(\int_{-\infty}^{+\infty} |f(t-\omega)|^p\, dt \right)^{1/p}. \end{aligned}$$

Now, making the variable change $\omega + \varphi = t$ and $d\varphi = dt$ with $\varphi \in \mathbb{R}$ in the second integral above, the result follows, that is,

$$\|f * g\|_{L^p(\mathbb{R})} \le \|f\|_{L^p(\mathbb{R})} \|g\|_{L^1(\mathbb{R})}.$$

Remark 3.2.1. The Properties 3.2.1 and 3.2.3 given above can be found in Theorem 3 of (Butzer and Jansche, 1997, pp. 339-340), including the proof of Property 3.2.3. The Properties 3.2.2-3.2.4 can be found in Theorem 2.10 of (Altın, 2011, pp. 106-107), including the proof of Property 3.2.4. The Properties 3.2.3 and 3.2.6 given above can be found in Theorem 6.1 of (Mamedov, 1991, pp. 40-41), including their proofs. Property 3.2.7 can be found in Theorem 1.3 of (Stein and Weiss, 1971, p. 3) with its proof. The Properties 3.2.3 and 3.2.5 can be found with their proofs in (Debnath and Bhatta, 2007, pp. 346-347). The written proofs (or clarifications) for properties discussed above, which are directly based on definitions of the concepts, can be seen and compared in the cited sources in this remark.
Now, we give some examples.

Example 3.2.1. Consider the functions $f, g: \mathbb{R}_+ \to \mathbb{C}$ defined by $f(\omega) = e^{-1/\omega^2}$ and $g(\omega) = \omega^2$. Their Mellin convolution product is obtained as follows:

$$
\begin{aligned}
(f *_M g)(t) &= \int_0^{+\infty} f\left(\frac{t}{\omega}\right) g(\omega) \frac{d\omega}{\omega} \\
&= \int_0^{+\infty} e^{-1/\left(\frac{t}{\omega}\right)^2} \omega^2 \frac{d\omega}{\omega} \\
&= \int_0^{+\infty} e^{-\left(\frac{\omega^2}{t^2}\right)} t\, dt.
\end{aligned}
$$

Let $u = \omega^2, \quad du = 2\omega\, d\omega$. We have

$$
\begin{aligned}
(f *_M g)(t) &= \frac{1}{2}\int_0^{+\infty} e^{\frac{-u}{t^2}} du \\
&= \frac{1}{2}\left[\frac{e^{\frac{-u}{t^2}}}{\frac{-1}{t^2}}\right]_0^{+\infty} \\
&= \frac{-t^2}{2}[0-1] \\
&= \frac{t^2}{2}.
\end{aligned}
$$

On the other hand, computations of $(g *_M f)$ leads to the following result:

$$
\begin{aligned}
(g *_M f)(t) &= \int_0^{+\infty} f(\omega)\, g\left(\frac{t}{\omega}\right) \frac{d\omega}{\omega} \\
&= \int_0^{+\infty} e^{-1/\omega^2} \frac{t^2}{\omega^2} \frac{d\omega}{\omega}.
\end{aligned}
$$

Let $u = \frac{-1}{\omega^2}, \quad du = \frac{2}{\omega^3}\, d\omega$. We have

$$(g *_M f)(t) = \frac{t^2}{2} \int_{-\infty}^{0} e^u \, du$$

$$= \frac{t^2}{2} \left[e^u \right|_{-\infty}^{0}$$

$$= \frac{t^2}{2} \left[1 - 0 \right]$$

$$= \frac{t^2}{2}.$$

Here, $t \in \mathbb{R}_+$. This result illustrates the commutativity of the classical Mellin convolution product.

Example 3.2.2. Consider the functions $f, g: \mathbb{R} \to \mathbb{C}$ defined by $f(t) = e^{2t}$ and $g(t) = e^{-t^2}$. Their classical convolution product is obtained as follows:

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-t)g(t) \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{+\infty} e^{2(x-t)} \, e^{-t^2} \, dt \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{+\infty} e^{-2t+2x} \, e^{-t^2} \, dt \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{+\infty} e^{2x} \, e^{-t^2-2t} \, dt \right)$$

$$= \frac{e^{2x}}{\sqrt{2\pi}} \left(\int_{-\infty}^{+\infty} e^{-t^2-2t} \, dt \right)$$

$$= \frac{e^{2x}}{\sqrt{2\pi}} \left(\int_{-\infty}^{+\infty} e^{1-(t+1)^2} \, dt \right).$$

Making the variable change $u = t + 1$ with $du = dt$, we have

$$\frac{e^{2x}}{\sqrt{2\pi}}\left(\int_{-\infty}^{+\infty} e^{1-(u)^2}\ du\right) = \frac{e^{2x+1}}{\sqrt{2}}.$$

On the other hand, computations of $(g * f)$ leads to the following result:

$$\begin{aligned}(g * f)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\,t)g(x-t)\,dt \\ &= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{+\infty} e^{2t}\, e^{-(x-t)^2}\ dt\right) \\ &= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{+\infty} e^{2t-(x-t)^2}\ dt\right).\end{aligned}$$

Making similar operations as above, we obtain:

$$\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{+\infty} e^{2t-(x-t)^2}\ dt\right) = \frac{e^{2x+1}}{\sqrt{2}}.$$

This result illustrates the commutativity of the classical convolution product.

Example 3.2.3. Consider the functions $f, g\colon \mathbb{R}_+ \to \mathbb{C}$ defined by $f\,(\omega) = e^{-\omega^2}$ and $g\,(\omega) = \omega$. In this case, $f \odot g$ is computed as follows:

$$\begin{aligned}(f \odot g)(t) &= \int_0^{+\infty} f(\omega t)g(\omega)\,d\omega, \\ &= \int_0^{+\infty} e^{-\omega^2 t^2}\,\omega d\omega.\end{aligned}$$

Let $u = \omega^2$ and $du = 2\omega\ d\omega$. We have

$$= \frac{1}{2}\int_0^{+\infty} e^{-t^2u}\, du$$

$$= \frac{1}{2}\left[\frac{e^{-t^2u}}{-t^2}\right|_0^{+\infty}$$

$$= \frac{-1}{2t^2}\,[0-1] = \frac{1}{2t^2}.$$

Next, the computation of $g \circledcirc f$ leads to the following result:

$$(g \circledcirc f)(t) = \int_0^{+\infty} f(\omega)g(\omega t)\, d\omega$$

$$= \int_0^{+\infty} e^{-\omega^2}\, \omega t d\omega.$$

Let $u = \omega^2$ and $du = 2\omega\, d\omega$. We have

$$= \frac{t}{2}\int_0^{+\infty} e^{-u}\, du$$

$$= \frac{t}{2}[-e^{-u}|_0^{+\infty}$$

$$= \frac{t}{2}\,[0+1] = \frac{t}{2}.$$

Here, $t \in \mathbb{R}_+$. In view of these results, it is easy to see that the operation is not commutative. For formal proof, see (Debnath and Bhatta, 2007).

3.3. MELLIN-TYPE LINEAR CONVOLUTION OPERATORS

Definition 3.3.1. Let $\boldsymbol{\Lambda_M} \neq \emptyset$ be a non-empty index set consisting of positive parameters σ with $\sigma \to +\infty$. The family $\{\mathbf{H}_\sigma\}_{\sigma \in \boldsymbol{\Lambda_M}} \subset L_0^1(\mathbb{R}_+)$ (i.e., $L_r^1(\mathbb{R}_+),\ r = 0$) is called a **kernel** on $L_0^1(\mathbb{R}_+)$ provided that

- $\int_0^{+\infty} \mathbf{H}_\sigma(u) \frac{du}{u} = 1$ for $\sigma > 0$

  and

- For all $\sigma \in \mathbf{\Lambda}_{\boldsymbol{M}}$, there is a real number $K > 0$ such that $\|\mathbf{H}_\sigma\|_{L_0^1(\mathbb{R}_+)} \leq K$.

If, in particular, $\{\mathbf{H}_\sigma\}_{\sigma \in \mathbf{\Lambda}_{\boldsymbol{M}}}$ verifies the following condition:

- For each fixed number ϑ satisfying $0 < \vartheta < 1$, one has

$$\lim_{\sigma \to +\infty} \int_{|1-\,y| \geq \vartheta} |\mathbf{H}_\sigma(u)| \frac{du}{u} = 0, \qquad y \in \mathbb{R}_+,$$

then the kernel family $\{\mathbf{H}_\sigma\}_{\sigma \in \mathbf{\Lambda}_{\boldsymbol{M}}}$ is named as a (local) **approximate identity** in the sense of Mellin (Butzer and Jansche, 1997).

Definition 3.3.2. The family of operators $\{T_\sigma\}_{\sigma > 0}$ with $T_\sigma : L_0^1(\mathbb{R}_+) \to L_0^1(\mathbb{R}_+)$ defined as:

$$(T_\sigma f)(x) = \int_0^{+\infty} f\left(\frac{x}{u}\right) \mathbf{H}_\sigma(u) \frac{du}{u}, \quad x \in \mathbb{R}_+ \tag{3.3.1}$$

is called a (local) **Mellin convolution type singular integral** provided that $\{\mathbf{H}_\sigma\}_{\sigma \in \mathbf{\Lambda}_{\boldsymbol{M}}} \subset L_0^1(\mathbb{R}_+)$ verifies the kernel assumptions (Butzer and Jansche, 1997).

Definition 3.3.3. Let $\mathbf{\Lambda} \neq \emptyset$ be a non-empty index set consisting of positive parameters σ with $\sigma \to +\infty$. The family $\{\mathbf{K}_\sigma\}_{\sigma \in \mathbf{\Lambda}}$ is called a **kernel** on $L^1(\mathbb{R})$ provided that:

- $\mathbf{K}_\sigma \in L^1(\mathbb{R})$ for each $\sigma \in \mathbf{\Lambda}$

  and

- $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathbf{K}_\sigma(t)\,dt = 1$ for $\sigma > 0$.

If it also verifies the following conditions:

- For all $\sigma \in \mathbf{\Lambda}$, there is a positive real number $\mathbf{N} > 0$ such that $\|\mathbf{K}_\sigma\|_{L^1(\mathbb{R})} \le \mathbf{N}$,
- For each fixed number ϑ satisfying $0 < \vartheta < \infty$, one has

$$\lim_{\sigma\to+\infty} \int\limits_{\vartheta\le|t|} |\mathbf{K}_\sigma(t)|\,dt = 0$$

and

$$\lim_{\sigma\to+\infty}\left[\sup_{\vartheta\le|t|}|\mathbf{K}_\sigma(t)|\right] = 0,$$

then the kernel family $\{\mathbf{K}_\sigma\}_{\sigma\in\mathbf{\Lambda}}$ is named as an **approximate identity** (Butzer and Nessel, 1971); see also (Gadjiev, 1998).

Definition 3.3.4. The family of operators $\{U_\sigma\}_{\sigma>0}$ with $U_\sigma: L^1(\mathbb{R}) \to L^1(\mathbb{R})$ defined as:

$$(U_\sigma f)(x) = \int\limits_{-\infty}^{+\infty} f(x-u)\mathbf{K}_\sigma(u)\,du, \quad x \in \mathbb{R} \tag{3.3.2}$$

is called a **convolution type singular integral** provided that $\{\mathbf{K}_\sigma\}_{\sigma\in\mathbf{\Lambda}} \subset L^1(\mathbb{R})$ verifies the kernel assumptions (Butzer and Nessel, 1971).

Remark 3.3.1. The following operators are non-classical versions of the operators defined in respectively in equations (3.3.1) and (3.3.2):

$$(T_\sigma^* f)(x) = \int\limits_{0}^{+\infty} f(u)\mathbf{H}_\sigma^*(ux^{-1})\frac{du}{u},$$

where $x, u \in \mathbb{R}_+$, $\sigma > 0$ and $\sigma \to +\infty$ (Bardaro and Mantellini, 2007)
and

$$(U_\sigma^* f)(x) = \int_{-\infty}^{+\infty} f(u)\mathbf{K}_\sigma^*(u-x)\,du,$$

where $x, u \in \mathbb{R}$, $\sigma > 0$ and $\sigma \to +\infty$ (Hacısalihoğlu and Hacıyev, 1995). The kernels here are also of type approximate identity.

Example 3.3.1. Butzer and Jansche (1997) gave the definition of Mellin-Gauss-Weierstrass kernel $\mathbf{H}_\sigma$ as follows:

$\mathbf{H}_\sigma : \mathbb{R}_+ \to \mathbb{R}$ is defined by $\mathbf{H}_\sigma(x) = \frac{\sigma}{\sqrt{4\pi}} e^{-\left(\frac{\sigma}{2}\ln(x)\right)^2}$, $\sigma > 0$.

Now, we will show that $\mathbf{H}_\sigma \in L_0^1(\mathbb{R}_+)$ as follows:

$$\|\mathbf{H}_\sigma\|_{L_0^1(\mathbb{R}_+)} = \int_0^{+\infty} \left|\frac{\sigma}{\sqrt{4\pi}} e^{-\left(\frac{\sigma}{2}\ln(x)\right)^2}\right| \frac{1}{x}\,dx\,.$$

Making the variable change $u = \frac{\sigma}{2}\ln(x)$ with $du = \frac{\sigma}{2x}\,dx$, we have

$$\|\mathbf{H}_\sigma\|_{L_0^1(\mathbb{R}_+)} = \frac{2}{\sqrt{4\pi}} \int_0^{+\infty} e^{-u^2}\,du = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1.$$

Proofs of all properties concerning being the approximate identity of this kernel in a more general fashion can be found in (Butzer and Jansche, 1997).

Example 3.3.2. The following versions of Gauss-Weierstrass operators:

$$G_\sigma(g;x) = \frac{\sigma}{\sqrt{\pi}} \int_{-\infty}^{+\infty} g(t+x)\, e^{-(\sigma t)^2} dt, \tag{3.3.3}$$

where $x, t \in \mathbb{R}$, $\sigma \in \mathbb{N}$ and $\sigma \to +\infty$ (Gadjiev, 1998),
and

$$W_\sigma(g; x) = \frac{\sqrt{\sigma}}{\sqrt{4\pi}} \int_{-\infty}^{+\infty} g(x-t)\, e^{\frac{-t^2\sigma}{4}} dt, \tag{3.3.4}$$

where $x, t \in \mathbb{R}$, $\sigma > 0$ and $\sigma \to +\infty$, or by replacing σ by $\frac{1}{\rho}$

$$W_\rho(g; x) = \frac{1}{\sqrt{4\pi\rho}} \int_{-\infty}^{+\infty} g(x-t)\, e^{\frac{-t^2}{4\rho}} dt, \tag{3.3.5}$$

where $x, t \in \mathbb{R}$, $\rho > 0$ and $\rho \to 0^+$ (Butzer and Nessel, 1971), have approximate identity kernels as treated in the works cited respectively.

PART 4

MELLIN-TYPE NONLINEAR CONVOLUTION OPERATORS

In this part, we give an overview of some results from the literature and prove a result concerning pointwise convergence of Mellin-type nonlinear m-singular integral operators at generalized Mellin $m-p-$Lebesgue points.

4.1. AN OVERVIEW OF RESULTS

Musielak (1983) studied the convergence properties of the following convolution-type nonlinear integral operators:

$$T_\sigma(g;x)=\int_a^b K_\sigma\big(t-x,g(t)\big)\,dt, \tag{4.1.1}$$

where $x\in\mathbb{R}$, σ is the element of a non-empty index set Ω and the kernel function K_σ, $K_\sigma\colon[a,b)\times\mathbb{R}\to\mathbb{R}$, satisfies some conditions. In order to overcome the nonlinearity problem occurring in the operators of type (4.1.1), Musielak (1983) stipulated that K_σ must satisfy the following Lipschitz condition:

$$|K_\sigma(t,u)-K_\sigma(t,u^*)|\le L_\sigma(t)|u-u^*|,$$

for $u,u^*\in\mathbb{R}$, $t\in[a,b)$ and $\sigma\in\Omega$ provided that such a non-negative function L_σ defined on $[a,b)$ exists with some additional features defined on it.

The operators of type (4.1.1) have been studied in different directions (see, for example, (Swiderski and Wachnicki, 2000; Angeloni and Vinti, 2006). Also, a large amount of information can be found in (Bardaro et al., 2003).

The linear convolution operators in different directions were studied in, for example, (Natanson, 1960; Taberski, 1962; Gadjiev, 1968; Butzer and Nessel, 1971; Stein and Weiss, 1971; Hacısalihoğlu and Hacıyev, 1995; Gadjiev, 1998; Anastassiou and Gal., 2000).

Bardaro and Mantellini (2006) studied the following Mellin-type nonlinear integral operators:

$$T_\sigma(g;x) = \int_0^{+\infty} K_\sigma\left(\frac{t}{x}, g(t)\right)\frac{dt}{t} = \int_0^{+\infty} K_\sigma\big(t, g(xt)\big)\frac{dt}{t}, \tag{4.1.2}$$

where $x, t \in \mathbb{R}_+$, $\sigma \in \mathbb{R}_+$ and the kernel function K_σ, $K_\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, satisfies some conditions. In this work, the authors obtained the pointwise convergence of the operators at Lebesgue points of the function g as $\sigma \to +\infty$ under suitable assumptions satisfied by K_σ. The operators of type (4.1.2) have been studied in different directions (see, for example, (Bardaro and Mantellini, 2007; Angeloni and Vinti 2014)). Also, Bardaro et al. (2011) studied the more generalized form of the operators of type (4.1.2). Also, a large amount of information can be found in (Bardaro et al., 2003).

Mellin-type linear convolution operators in different directions were studied in, for example, (Mamedov, 1991; Butzer and Jansche, 1997; Butzer and Jansche, 1998; Bardaro and Mantellini, 2007; Bardaro and Mantellini, 2011; Angeloni and Vinti, 2015; Fard and Zainuddin, 2016).

Bardaro et al. (2013) defined and considered the following Mellin-type nonlinear m −singular integral operators:

$$T_\sigma^{[m]}(g;z) = \int_0^{+\infty} K_\sigma\left(t, \sum_{k=1}^{m} \binom{m}{k} (-1)^{k-1}\, g(zt^k)\right)\frac{dt}{t},$$

where $z, t \in \mathbb{R}_+$, σ is the element of a non-empty index set Ω, m is a fixed positive integer and the kernel function K_σ, $K_\sigma: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, satisfies some conditions. In this work, the authors obtained the pointwise convergence of the operators at $m-$Lebesgue points of the function $g \in \mathcal{L}^1(\mathbb{R}_+)$ as $\sigma \to \sigma_0$, where σ_0 is an accumulation point of Ω with respect to the topology defined on it, under suitable assumptions satisfied by K_σ.

Taking $m = 1$ and $\sigma_0 = +\infty$ with $\sigma > 0$, the setting of the operators defined in equation (4.1.2) is obtained.

Mellin-type linear $m-$singular integral operators were studied widely by Mamedov (1991), presenting various results. Mamedov (1963) also studied the pointwise convergence of linear $m-$singular integral operators in the space $L^P(\mathbb{R})$ with $1 \leq p < +\infty$. Some studies related to this kind of generalization of the integral operators can be given as (Anastassiou and Gal., 2000; Ibrahimov and Jafarova, 2012; Karslı, 2014; Yılmaz, 2014; Yıldırım, 2019).

4.2. SOME POINTWISE CONVERGENCE THEOREMS

Two results from the literature are reviewed in detail, and in view of them, a pointwise convergence result is proved in the sequel.

We start by reviewing Theorem 30.1 in (Mamedov, 1991).

Mamedov (1991) defined the following Mellin-type linear $m-$singular integral operators:

$$\mathrm{H}_\sigma^{[m]}(g; z) = \int_0^{+\infty} \left[\sum_{k=1}^{m} \binom{m}{k} (-1)^{k-1} g\left(\frac{z}{t^k}\right)\right] \mathrm{K}_\sigma(t) \frac{dt}{t},$$

where $z, t \in \mathbb{R}_+$, σ is the element of a non-empty index set Ω, m is a fixed positive integer, the accumulation point of Ω is denoted by σ_0 and the kernel function K_σ, $\mathrm{K}_\sigma : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{0\}$, satisfies some conditions.

Assuming K_σ satisfies the following conditions:

i. $\int_0^{+\infty} \mathrm{K}_\sigma(t) \frac{dt}{t} = 1, \; t \in \mathbb{R}_+;$

ii. $\lim_{\sigma \to \sigma_0} [\sup_{0<t<1-\delta} \mathrm{K}_\sigma(t)] = \lim_{\sigma \to \sigma_0} [\sup_{1+\delta<t<+\infty} \mathrm{K}_\sigma(t)] = 0$ for each fixed $\delta \in (0,1)$ and $\lim\limits_{\sigma \to \sigma_0} \left[\int_0^{1-\delta} \mathrm{K}_\sigma(t) \frac{dt}{t}\right] = \lim\limits_{\sigma \to \sigma_0} \left[\int_{1+\delta}^{+\infty} \mathrm{K}_\sigma(t) \frac{dt}{t}\right] = 0$ for each fixed $\delta \in (0,1)$;

iii. There exists a positive real number M such that $\|\mathrm{K}_\sigma\|_{\mathcal{L}^1(\mathbb{R}_+)} \le M$, where M is independent of $\sigma \in \Omega$ (this condition is equivalent to condition (i) for this kernel since it is non-negative);

iv. $\mathrm{K}_\sigma(t)$ is non-decreasing on $(0,1)$ and non-increasing on $(1,+\infty)$ with respect to t for each fixed $\sigma \in \Omega$;

v. There exists a positive real number η such that $\lim\limits_{\sigma \to \sigma_0} [\chi_\sigma] = 0$, where
$\chi_\sigma = \int_{1-q}^{1+q} \mathrm{K}_\sigma(t) |\ln(t)|^\alpha \frac{dt}{t}$ for each fixed $q \in (0,1)$ and $0 < \alpha \le \eta$,

Mamedov (1991) proved that

$$\left|\mathrm{H}_\sigma^{[m]}(g; z) - g(z)\right| = o\left((\chi_\sigma)^{\frac{\alpha}{\eta}}\right) \;\; (\sigma \to \sigma_0),$$

where $g \in \mathcal{L}^p(\mathbb{R}_+)$ with $1 \le p < +\infty$, holds at each point $z \in \mathbb{R}_+$ for which the following relations hold:

$$\int_{1-h}^{1} \left|\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} g(zt^k)\right|^p \frac{dt}{t} = o(|\ln(1-h)|^+) \;\; (h \to 0^+)$$

and

$$\int_{1}^{1+h} \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} g(zt^k) \right|^p \frac{dt}{t} = o(|\ln(1+h)|^+) \ \ (h \to 0^+)$$

provided that:

I. $\sup_{0<t<1-\delta} \mathrm{K}_\sigma(t) = o\left((\chi_\sigma)^{\frac{\alpha}{\eta}}\right)$ and $\sup_{1+\delta<t<+\infty} \mathrm{K}_\sigma(t) = o\left((\chi_\sigma)^{\frac{\alpha}{\eta}}\right)$ $(\sigma \to \sigma_0)$ for each fixed $\delta \in (0,1)$;

II. $\int_0^{1-\delta} \mathrm{K}_\sigma(t) \frac{dt}{t} = o\left((\chi_\sigma)^{\frac{\alpha}{\eta}}\right)$ and $\int_{1+\delta}^{+\infty} \mathrm{K}_\sigma(t) \frac{dt}{t} = o\left((\chi_\sigma)^{\frac{\alpha}{\eta}}\right)$ $(\sigma \to \sigma_0)$ for each fixed $\delta \in (0,1)$.

Here, "o" stands for Landau's little o notation.

Now, we summarize Theorem 3.1 in (Bardaro et al., 2013).

Let Ω be a non-empty index set associated with a topology. Here, σ denotes the element of Ω. The accumulation point of Ω is denoted by σ_0 with respect to the indicated topology. Recall that Bardaro et al. (2013) considered the following Mellin-type nonlinear m −singular integral operators:

$$T_\sigma^{[m]}(g;z) = \int_0^{+\infty} K_\sigma\left(t, \sum_{k=1}^{m} \binom{m}{k} (-1)^{k-1} g(zt^k)\right) \frac{dt}{t}, \qquad (4.2.1)$$

where $z, t \in \mathbb{R}_+$, m is a fixed positive integer and K_σ satisfies some conditions, that is, the family K, which is called the kernel of the family of the operators $\left\{T_\sigma^{[m]}\right\}_{\sigma\in\Omega}$, consists of the functions $K_\sigma:\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ with $K_\sigma(t,0) = 0$ for every $t \in \mathbb{R}_+$, $\sigma \in \Omega$ and K_σ is integrable as a function of its first variable over $\mathbb{R}_+$ for all values of its second variable and for each $\sigma \in \Omega$ with respect to the Haar measure μ defined on Borel sets B on $\mathbb{R}_+$ with $\mu(B) = \int_B t^{-1}dt = \int_B \frac{dt}{t}$. The authors also denote the set of all functions g for which $T_\sigma^{[m]}$ is well-defined by $\mathrm{Dom}\left(T_\sigma^{[m]}\right)$, the family of all neighbourhoods of number 1 in $\mathbb{R}_+$ by $N(1)$ and $\max_{k=1,\dots,m}\binom{m}{k}$ by m_0.

In view of these, the authors proved that:

$$\lim_{\sigma\to\sigma_0}\left|T_\sigma^{[m]}(g;z)-g(z)\right|=0,$$

where $g\in \mathrm{Dom}\left(T_\sigma^{[m]}\right)\cap\mathcal{L}^1(\mathbb{R}_+)$ with $\psi(m_0|g|)\in\mathcal{L}^1(\mathbb{R}_+)$ and ψ, $\psi\colon\mathbb{R}_+\cup\{0\}\to\mathbb{R}_+\cup\{0\}$, is a continuous, non-decreasing and concave function on $\mathbb{R}_+\cup\{0\}$ with $\psi(0)=0$ and $\psi(u)>0$ for $u>0$, holds at each point $z\in\mathbb{R}_+$ for which the following relation holds:

$$\lim_{z\to1}\left|\frac{1}{\ln(z)}\int_1^z\left|\sum_{k=0}^m\binom{m}{k}(-1)^{m-k}\,g(zt^k)\right|\frac{dt}{t}\right|=0$$

(see Definition 2 of the same paper) provided that the kernel function K_σ, which is described above, satisfies the following conditions:

A. There exists a function $L_\sigma\colon\mathbb{R}_+\to\mathbb{R}_+\cup\{0\}$ with $L_\sigma\in\mathcal{L}^1(\mathbb{R}_+)$ and such that the ψ −Lipschitz condition
$|K_\sigma(t,u)-K_\sigma(t,u^*)|\le L_\sigma(t)\psi(|u-u^*|)$
hold for every $u,u^*\in\mathbb{R}$, $t\in\mathbb{R}_+$ and for each $\sigma\in\Omega$;

B. $\sup_{\sigma\in\Omega}\int_0^{+\infty}L_\sigma(y)\frac{dy}{y}=A_1$, where $y\in\mathbb{R}_+$ and A_1 is a certain positive constant;

C. $\lim_{\sigma\to\sigma_0}\left|\int_0^{+\infty}K_\sigma(y,u)\frac{dy}{y}-u\right|=0$ for every $u\in\mathbb{R}$, where $y\in\mathbb{R}_+$;

D. $\lim_{\sigma\to\sigma_0}\int_{\mathbb{R}_+\setminus\left[\frac{1}{\xi},\xi\right]}L_\sigma(y)\frac{dy}{y}=0$ for every $\xi>1$ with $\left[\frac{1}{\xi},\xi\right]\in N(1)$;

E. $\lim_{\sigma\to\sigma_0}\sup_{y\in\mathbb{R}_+\setminus\left[\frac{1}{\xi},\xi\right]}[L_\sigma(y)]=0$ for every $\xi>1$ with $\left[\frac{1}{\xi},\xi\right]\in N(1)$;

F. There exists a positive number $\xi_0 > 1$ for which $L_\sigma(y)$ is non-decreasing on $(\frac{1}{\xi_0}, 1]$ and non-increasing on $[1, \xi_0)$ with respect to y for each $\sigma \in \Omega$.

Remark 4.2.1. Now, we recall here the useful identities and a relation that are classically used in the proofs. There hold the following inequalities (see, for example, (Grinstead and Snell, 1999)):

$$\sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} = 1 \tag{4.2.2}$$

and

$$\sum_{k=0}^{m} \binom{m}{k} = 2^m, \tag{4.2.3}$$

for a certain positive integer m. Also, the inequality $(a+b)^p \leq 2^p(a+b)^p$ holds for $1 \leq p < \infty$ and non-negative real numbers a and b (see, for example, (Rudin, 1987)).

Now, based on the above-reviewed theorems, we will prove a pointwise convergence result for the operators defined in equation (4.2.1) which are considered and studied by Bardaro et al. (2013).

Definition 4.2.1. Let $g \in \mathcal{L}^p(\mathbb{R}_+)$ with $p \geq 1$. A point $z \in \mathbb{R}_+$ at which

$$\lim_{h \to 0^+} \frac{1}{(\ln(1-h))^{1+\alpha}} \int_{1-h}^{1} \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} g(zt^k) \right|^p \frac{dt}{t} = 0 \tag{4.2.4}$$

and

$$\lim_{h\to 0^+} \frac{1}{(\ln(1+h))^{1+\alpha}} \int_1^{1+h} \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} g(zt^k) \right|^p \frac{dt}{t} = 0 \qquad (4.2.5)$$

where $0 < h < 1$ and $0 \le \alpha < \eta$, for a certain real number $\eta > 0$ and a fixed positive integer m hold is called generalized Mellin $m - p$ −Lebesgue point of g (Mamedov, 1991).

For $m = 1$ and $\alpha = 0$ the above definition turns out to be the definition of Mellin p −Lebesgue point given by the same author. Mamedov (1991) also showed that for the case $\alpha \ge mp$, g corresponds to the function which is identically zero.

Example 4.2.1. Let $f(\rho) = \begin{cases} \rho, & if \quad \rho \in (0,3) \\ 0, & if \;\; \rho \in \mathbb{R}_+ \backslash (0,3) \end{cases}$ with $p = 1$ and $z = 1$.

Let us show that this point is Mellin Lebesgue point of f. Applying the definition for $p = 1$ and $m = 1$, we get

$$\lim_{h\to 0^+} \frac{1}{\ln(1+h)} \int_1^{1+h} \left| f\left(\frac{1}{\rho}\right) - 1 \right| \frac{d\rho}{\rho}$$

$$= \lim_{h\to 0^+} \frac{1}{\ln(1+h)} \int_1^{1+h} \left| \frac{1}{\rho} - 1 \right| \frac{d\rho}{\rho}$$

$$= \lim_{h\to 0^+} \frac{1}{\ln(1+h)} \int_1^{1+h} \left(1 - \frac{1}{\rho}\right) \frac{d\rho}{\rho}$$

$$= \lim_{h\to 0^+} \frac{1}{\ln(1+h)} \left[\ln\rho|_1^{1+h} + \frac{1}{\rho}\Big|_1^{1+h} \right]$$

$$= \lim_{h\to 0^+} \frac{1}{\ln(1+h)} \left[(\ln(1+h) - \ln(1)) + (\frac{1}{1+h} - 1) \right]$$

$$= \lim_{h\to 0^+} \frac{\ln(1+h) + \frac{1}{1+h} - 1}{\ln(1+h)}.$$

Here, using L'Hospital's theorem, we have

$$\lim_{h\to 0^+}\frac{\ln(1+h\,)+\dfrac{1}{1+h}-1}{\ln(1+h)} = \lim_{h\to 0^+}\frac{\dfrac{1}{1+h}-\dfrac{1}{(1+h)^2}}{\dfrac{1}{1+h}}$$

$$= \frac{\dfrac{1}{1+0}-\dfrac{1}{(1+0)^2}}{\dfrac{1}{1+0}}$$

$$= 0.$$

Verification of the relation (4.2.4) can be done in the same way.

Remark 4.2.2. In the following definition, the conditions and properties on K_σ given by Bardaro et al. (2013) and the conditions on K_σ given by Mamedov (1991) are combined as seen below. On the other hand, we use the Lipschitz condition on K_ρ in equation (4.1.1), which is given by Musielak (1983). That case is obtained from ψ −Lipschitz condition by choosing $\psi(u) = |u|$ (see (Bardaro et al., 2003; Bardaro and Mantellini, 2006). Also, in the forthcoming condition (f), N need not be zero as in (Mamedov, 1991) because we just obtain pointwise convergence; we do not evaluate the rate of convergence as in Theorem 30.1 reviewed above. We refer the reader to see the complete work of Mamedov (1991) since it covers many pointwise convergence results, and also (Gadjiev, 1968) for this kind of theorem concerning generalized Lebesgue point.

Definition 4.2.2. Let Ω be a non-empty index set associated with a topology. Here, σ denotes the element of Ω. The accumulation point of Ω is denoted by σ_0, that is, $\sigma \to \sigma_0$ (in particular, $\sigma \to +\infty$ for $\sigma_0 = +\infty$ with $\sigma > 0$).

The kernel function K_σ of the operator $T_\sigma^{[m]}$, which is defined as K_σ:$\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ for each fixed $\sigma \in \Omega$, verifies the following conditions:

a) For all $u \in \mathbb{R}$ and for each fixed $\sigma \in \Omega$, $K_\sigma(t,u) \in \mathcal{L}^1(\mathbb{R}_+)$ with respect to t and $K_\sigma(t,0) = 0$ for every $t \in \mathbb{R}_+$ and for each fixed $\sigma \in \Omega$;
b) There exists a non-negative function L_σ defined on $\mathbb{R}_+$ such that $L_\sigma \in \mathcal{L}^1(\mathbb{R}_+)$ and the Lipschitz condition

$|K_\sigma(t,u) - K_\sigma(t,u^*)| \le L_\sigma(t)|u-u^*|$

hold for every $u, u^* \in \mathbb{R}$, $t \in \mathbb{R}_+$ and for each fixed $\sigma \in \Omega$;

c) $\lim_{\sigma\to\sigma_0}[\sup_{0<t<1-\delta} L_\sigma(t)] = \lim_{\sigma\to\sigma_0}[\sup_{1+\delta<t<+\infty} L_\sigma(t)] = 0$ for each fixed $\delta \in (0,1)$ and $\lim\limits_{\sigma\to\sigma_0}\left[\int_0^{1-\delta} L_\sigma(t)\frac{dt}{t}\right] = \lim\limits_{\sigma\to\sigma_0}\left[\int_{1+\delta}^{+\infty} L_\sigma(t)\frac{dt}{t}\right] = 0$ for each fixed $\delta \in (0,1)$;

d) $\lim\limits_{\sigma\to\sigma_0}\left|\int_0^{+\infty} K_\sigma(t,u)\frac{dt}{t} - u\right| = 0$ for every $u \in \mathbb{R}$;

e) $L_\sigma(t)$ is non-decreasing on $(0,1)$ and non-increasing on $(1,+\infty)$ with respect to t for each fixed $\sigma \in \Omega$;

f) There is a positive real number η such that $\lim\limits_{\sigma\to\sigma_0}\left[\int_{1-\delta}^{1+\delta} L_\sigma(t)|\ln(t)|^\alpha\frac{dt}{t}\right] = N$, where N is a non-negative real number, for each fixed $\delta \in (0,1)$ and $0 \le \alpha < \eta$;

g) There is a positive real number M such that $\|L_\sigma\|_{\mathcal{L}^1(\mathbb{R}_+)} \le M$, where M is independent of $\sigma \in \Omega$.

Note that the operators $T_\sigma^{[m]}$ defined in equation (4.2.1) act on $\mathcal{L}^p(\mathbb{R}_+)$ with $p \ge 1$ in view of conditions (a, b, g) of Definition 4.2.2.

Theorem 4.2.1. Assume that K_σ satisfies the conditions in Definition 4.2.2 and $T_\sigma^{[m]}$ are as in equation (4.2.1). If $g \in \mathcal{L}^p(\mathbb{R}_+)$ with $1 \le p < +\infty$, then

$$\lim_{\sigma\to\sigma_0}\left|T_\sigma^{[m]}(g;z) - g(z)\right| = 0$$

holds at each generalized Mellin $m-p$ −Lebesgue point $z \in \mathbb{R}_+$ of g for which equations (4.2.4) and (4.2.5) hold.

Proof. We mainly follow the steps of Theorem 3.1 in (Bardaro et al., 2013) and Theorems 25.2 and 30.1 (Mamedov, 1991), which are based on classical proof schemes.

The proof for the case $p = 1$ is as follows:

$$\left|T_\sigma^{[m]}(g;z)-g(z)\right| = \left|\int_0^{+\infty} K_\sigma\left(t,\sum_{k=1}^{m}\binom{m}{k}(-1)^{k-1}g(zt^k)\right)\frac{dt}{t}-g(z)\right|.$$

Adding and subtracting the expression $\int_0^{+\infty} K_\sigma\left(t,\sum_{k=1}^{m}\binom{m}{k}(-1)^{k-1}g(z)\right)\frac{dt}{t}$ to the expression inside the absolute value on the right-hand side above, we have

$$\left|\int_0^{+\infty} K_\sigma\left(t,\sum_{k=1}^{m}\binom{m}{k}(-1)^{k-1}g(zt^k)\right)\frac{dt}{t}-g(z)\right.$$
$$\left.\mp\int_0^{+\infty} K_\sigma\left(t,\sum_{k=1}^{m}\binom{m}{k}(-1)^{k-1}g(z)\right)\frac{dt}{t}\right|.$$

Using triangle inequality, we get

$$\left|T_\sigma^{[m]}(g;z)-g(z)\right|$$
$$\leq\left|\int_0^{+\infty} K_\sigma\left(t,\sum_{k=1}^{m}\binom{m}{k}(-1)^{k-1}g(zt^k)\right)\frac{dt}{t}\right.$$
$$\left.-\int_0^{+\infty} K_\sigma\left(t,\sum_{k=1}^{m}\binom{m}{k}(-1)^{k-1}g(z)\right)\frac{dt}{t}\right|$$
$$+\left|\int_0^{+\infty} K_\sigma\left(t,\sum_{k=1}^{m}\binom{m}{k}(-1)^{k-1}g(z)\right)\frac{dt}{t}-g(z)\right|$$

$=: A_1(\sigma)+A_2(\sigma).$

First, we show that $A_1(\sigma)\to 0$ as $\sigma\to\sigma_0$.

Using the Lipschitz condition, we get

$$A_1(\sigma) \leq \int_0^{+\infty} L_\sigma(t) \left| \sum_{k=1}^{m} \binom{m}{k} (-1)^{k-1} g(zt^k) - g(z) \right| \frac{dt}{t}$$

$$= \int_0^{+\infty} L_\sigma(t) \left| \sum_{k=1}^{m} \binom{m}{k} (-1)^{k-1} g(zt^k) - \sum_{k=1}^{m} \binom{m}{k} (-1)^{k-1} g(z) \right| \frac{dt}{t}$$

$$= \int_0^{+\infty} L_\sigma(t) \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{k-1} g(zt^k) \right| \frac{dt}{t}.$$

Since

$$\left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{k-1} g(zt^k) \right| = \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{1-k} g(zt^k) \right|$$

$$= \left| (-1)^{2m} \sum_{k=0}^{m} \binom{m}{k} (-1)^{1-k} g(zt^k) \right|$$

$$= \left| (-1)^{m+1} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} g(zt^k) \right|$$

$$= \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} g(zt^k) \right|,$$

we have

$$\int_0^{+\infty} L_\sigma(t) \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{k-1} g(zt^k) \right| \frac{dt}{t} = \int_0^{+\infty} L_\sigma(t) \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} g(zt^k) \right| \frac{dt}{t}$$

$$= A_{11}(\sigma).$$

Let $\delta \in (0,1)$. We can split the integral $A_{11}(\sigma)$ into four terms as follows:

$$A_{11}(\sigma) = \int_0^{1-\delta} L_\sigma(t) \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} g(zt^k) \right| \frac{dt}{t}$$

$$+\int_{1-\delta}^{1} L_\sigma(t)\left|\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\,g(zt^k)\right|\frac{dt}{t}$$

$$+\int_{1}^{1+\delta} L_\sigma(t)\left|\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\,g(zt^k)\right|\frac{dt}{t}$$

$$+\int_{1+\delta}^{+\infty} L_\sigma(t)\left|\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\,g(zt^k)\right|\frac{dt}{t}$$

$$=: A_{111}(\sigma)+A_{112}(\sigma)+A_{113}(\sigma)+A_{114}(\sigma).$$

In view of equations (4.2.4) and (4.2.5) and using the usual definition of limit, we can write that for a given $\epsilon > 0$, there exists $\delta_* > 0$ such that

$$\left|\left(\frac{1}{(\ln(1-h))^{1+\alpha}}\int_{1-h}^{1}\left|\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\,g(zt^k)\right|\frac{dt}{t}\right)-0\right|\le\epsilon \qquad (4.2.6)$$

and

$$\left|\left(\frac{1}{(\ln(1+h))^{1+\alpha}}\int_{1}^{1+h}\left|\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\,g(zt^k)\right|\frac{dt}{t}\right)-0\right|\le\epsilon \qquad (4.2.7)$$

hold provided that $0 < h \le \delta < \delta_*$.

The relations (4.2.6) and (4.2.7) can be written respectively as follows:

$$\int_{1-h}^{1}\left|\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\,g(zt^k)\right|\frac{dt}{t}\le\epsilon|\ln(1-h)|^{1+\alpha} \qquad (4.2.8)$$

and

$$\int_{1}^{1+h} \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} g(zt^k) \right| \frac{dt}{t} \leq \epsilon |\ln(1+h)|^{1+\alpha} \tag{4.2.9}$$

for $\epsilon > 0$ and $0 < h \leq \delta < \delta_*$.

Now, we compute $A_{112}(\sigma)$ as follows:

In view of relation (4.2.8) and the auxiliary function:

$$F(t) := \int_{t}^{1} \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} g(zu^k) \right| \frac{du}{u},$$

we see that $|F(t)| \leq \epsilon |\ln(t)|^{1+\alpha}$ for $t \in [1-\delta, 1)$. Using $F(t)$ we obtain the following equality:

$$|A_{112}(\sigma)| = \left| \int_{1-\delta}^{1} L_\sigma(t) \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} g(zt^k) \right| \frac{dt}{t} \right|$$
$$= \left| \int_{1-\delta}^{1} L_\sigma(t) dF(t) \right|.$$

Since $L_\sigma(t)$ is the non-decreasing function of t on $[1-\delta, 1)$ for each fixed σ, it is of bounded variation and hence differentiable almost everywhere (see Corollary 2.1.1). Using integration by parts, we have

$$|A_{112}(\sigma)| = \left| \int_{1-\delta}^{1} L_\sigma(t) dF(t) \right|$$
$$= \left| [F(t) L_\sigma(t)]_{1-\delta}^{1} - \int_{1-\delta}^{1} F(t) dL_\sigma(t) \right|$$

$$= \left| [0 - F(1-\delta)L_\sigma(1-\delta)] - \int_{1-\delta}^{1} F(t)\, dL_\sigma(t) \right|$$

$$\leq \epsilon |\ln(1-\delta)|^{1+\alpha} L_\sigma(1-\delta) + \epsilon \left| \int_{1-\delta}^{1} |\ln(t)|^{1+\alpha} dL_\sigma(t) \right|.$$

Since $L_\sigma(t)$ is the non-decreasing function of t on $[1-\delta, 1)$, $dL_\sigma(t) \geq 0$ for each fixed σ. Therefore, we have

$$|A_{112}(\sigma)| \leq \left| \epsilon |\ln(1-\delta)|^{1+\alpha} L_\sigma(1-\delta) + \epsilon \int_{1-\delta}^{1} |\ln(t)|^{1+\alpha} dL_\sigma(t) \right|.$$

We will integrate this part: $\int_{1-\delta}^{1} |\ln(t)|^{1+\alpha}\, dL_\sigma(t)$.

Using integration by parts method, that is, letting $u = |\ln(t)|^{1+\alpha}$ with $du = (1+\alpha)|\ln(t)|^{\alpha} \frac{dt}{t}$ and $dv = dL_\sigma(t)$ with $v = L_\sigma(t)$, we have

$$\int_{1-\delta}^{1} |\ln(t)|^{1+\alpha}\, dL_\sigma(t)$$

$$= [|\ln(t)|^{1+\alpha}\, L_\sigma(t)]_{1-\delta}^{1} - (1+\alpha) \int_{1-\delta}^{1} L_\sigma(t) |\ln(t)|^{\alpha} \frac{dt}{t}$$

$$= [0 - |\ln(1-\delta)|^{1+\alpha}\, L_\sigma(1-\delta)] - (1+\alpha) \int_{1-\delta}^{1} L_\sigma(t) |\ln(t)|^{\alpha} \frac{dt}{t}.$$

Collecting all the terms, we get

$$|A_{112}(\sigma)| \leq \Bigg| \epsilon |\ln(1-\delta)|^{1+\alpha} L_\sigma(1-\delta) - |\ln(1-\delta)|^{1+\alpha} L_\sigma(1-\delta) - \epsilon(1+\alpha) \int_{1-\delta}^{1} L_\sigma(t) |\ln(t)|^\alpha \frac{dt}{t} \Bigg|.$$

Thus:

$$|A_{112}(\sigma)| \leq \epsilon(1+\alpha) \int_{1-\delta}^{1} L_\sigma(t) |\ln(t)|^\alpha \frac{dt}{t}.$$

In view of relation (4.2.9) and the auxiliary function:

$$H(t) \coloneqq \int_1^t \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} g(zu^k) \right| \frac{du}{u},$$

we see that $|H(t)| \leq \epsilon |\ln(t)|^{1+\alpha}$ for $t \in (1, 1+\delta]$. Using $H(t)$ we obtain the following equality:

$$|A_{113}(\sigma)| = \left| \int_1^{1+\delta} L_\sigma(t) \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} g(zt^k) \right| \frac{dt}{t} \right| = \left| \int_1^{1+\delta} L_\sigma(t) dH(t) \right|.$$

Since $L_\sigma(t)$ is the non-increasing function of t on $(1, 1+\delta]$ for each fixed σ, it is of bounded variation and hence differentiable almost everywhere (see Corollary 2.1.1). Using integration by parts, we have

$$|A_{113}(\sigma)| = \left| \int_1^{1+\delta} L_\sigma(t) dH(t) \right|$$

$$= \left| [H(t)L_\sigma(t)]_1^{1+\delta} - \int_1^{1+\delta} H(t) dL_\sigma(t) \right|$$

$$= \left| [H(1+\delta)L_\sigma(1+\delta) - 0] - \int_1^{1+\delta} H(t)\, dL_\sigma(t) \right|$$

$$\leq \epsilon |\ln(1+\delta)|^{1+\alpha} L_\sigma(1+\delta) + \epsilon \left| \int_1^{1+\delta} |\ln(t)|^{1+\alpha} dL_\sigma(t) \right|.$$

Since $L_\sigma(t)$ is the non-increasing function of t on $(1, 1+\delta]$, $dL_\sigma(t) \leq 0$ for each fixed σ. Therefore

$$|A_{113}(\sigma)| \leq \left| \epsilon |\ln(1+\delta)|^{1+\alpha} L_\sigma(1+\delta) - \epsilon \int_1^{1+\delta} |\ln(t)|^{1+\alpha} dL_\sigma(t) \right|.$$

We will integrate this part: $\int_1^{1+\delta} |\ln(t)|^{1+\alpha}\, dL_\sigma(t)$.

Using integration by parts method, that is, letting $u = |\ln(t)|^{1+\alpha}$ with $du = (1+\alpha)|\ln(t)|^\alpha \frac{dt}{t}$ and $dv = dL_\sigma(t)$ with $v = L_\sigma(t)$, we have

$$\int_1^{1+\delta} |\ln(t)|^{1+\alpha}\, dL_\sigma(t)$$

$$= [|\ln(t)|^{1+\alpha}\, L_\sigma(t)]_1^{1+\delta} - (1+\alpha) \int_1^{1+\delta} L_\sigma(t) |\ln(t)|^\alpha \frac{dt}{t}$$

$$= [|\ln(1+\delta)|^{1+\alpha}\, L_\sigma(1+\delta) - 0] - (1+\alpha) \int_1^{1+\delta} L_\sigma(t) |\ln(t)|^\alpha \frac{dt}{t}.$$

Collecting all the terms, we get

$$|A_{113}(\sigma)| \leq \epsilon (1+\alpha) \int_1^{1+\delta} L_\sigma(t) |\ln(t)|^\alpha \frac{dt}{t}.$$

Hence,

$$(|A_{112}(\sigma)| + |A_{113}(\sigma)|) \leq \epsilon(1+\alpha)\left\{\int_{1-\delta}^{1} L_\sigma(t)|\ln(t)|^\alpha \frac{dt}{t} + \int_{1}^{1+\delta} L_\sigma(t)|\ln(t)|^\alpha \frac{dt}{t}\right\}$$

$$= \epsilon(1+\alpha)\int_{1-\delta}^{1+\delta} L_\sigma(t)|\ln(t)|^\alpha \frac{dt}{t}.$$

Since $\int_{1-\delta}^{1+\delta} L_\sigma(t)|\ln(t)|^\alpha \frac{dt}{t}$ is bounded in the limit position for each fixed $\delta > 0$ and $0 \leq \alpha < \eta$ by condition (f) the result follows, that is,
$\lim_{\sigma\to\sigma_0} (|A_{112}(\sigma)| + |A_{113}(\sigma)|) = 0.$

Now, we will show that $A_{111}(\sigma) \to 0$ as $\sigma \to \sigma_0$.

Since

$$|A_{111}(\sigma)| = \left|\int_0^{1-\delta} L_\sigma(t)\left|\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k} g(zt^k)\right|\frac{dt}{t}\right|$$

$$= \left|\int_0^{1-\delta} L_\sigma(t)\left|\sum_{k=1}^{m}\binom{m}{k}(-1)^{m-k} g(zt^k) - g(z)\right|\frac{dt}{t}\right|$$

$$\leq \sum_{k=1}^{m}\binom{m}{k}\sup_{0<t<1-\delta} L_\sigma(t)\int_0^{+\infty}|g(zt^k)|\frac{dt}{t} + |g(z)|\int_0^{1-\delta} L_\sigma(t)\frac{dt}{t}$$

$$\leq \sum_{k=0}^{m}\binom{m}{k}\sup_{0<t<1-\delta} L_\sigma(t)\,\|g\|_{\mathcal{L}^1(\mathbb{R}_+)} + |g(z)|\int_0^{1-\delta} L_\sigma(t)\frac{dt}{t}$$

$$= 2^m \sup_{0<t<1-\delta} L_\sigma(t)\|g\|_{\mathcal{L}^1(\mathbb{R}_+)} + |g(z)|\int_0^{1-\delta} L_\sigma(t)\frac{dt}{t},$$

where $\sum_{k=0}^{m}\binom{m}{k}$ is equal to 2^m (see equation 4.2.3), and by condition (c), we see that $\lim_{\sigma\to\sigma_0}|A_{111}(\sigma)| = 0$. Here, by definition of norm $\int_0^{+\infty}|g(zt^k)|\frac{dt}{t}$ is equal to

$\|g\|_{\mathcal{L}^1(\mathbb{R}_+))}$ since making the variable change $zt^k = u$ with $kzt^{k-1}dt = du$ and in view of this $\frac{dt}{t} = \frac{du}{u}$, we have

$$\int_0^{+\infty} |g(zt^k)| \frac{dt}{t} = \int_0^{+\infty} |g(u)| \frac{du}{u} = \|g\|_{\mathcal{L}^1(\mathbb{R}_+)}.$$

Similarly, we will show that $A_{114}(\sigma) \to 0$ as $\sigma \to \sigma_0$.

Making similar evaluations, we have

$$\begin{aligned}
|A_{114}(\sigma)| &= \left| \int_{1+\delta}^{+\infty} L_\sigma(t) \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} g(zt^k) \right| \frac{dt}{t} \right| \\
&= \left| \int_{1+\delta}^{+\infty} L_\sigma(t) \left| \sum_{k=1}^{m} \binom{m}{k} (-1)^{m-k} g(zt^k) - g(z) \right| \frac{dt}{t} \right| \\
&\leq \sum_{k=0}^{m} \binom{m}{k} \sup_{1+\delta<t<+\infty} L_\sigma(t) \int_0^{+\infty} |g(zt^k)| \frac{dt}{t} + |g(z)| \int_{1+\delta}^{+\infty} L_\sigma(t) \frac{dt}{t}.
\end{aligned}$$

Here, by definition of norm $\int_0^{+\infty} |g(zt^k)| \frac{dt}{t}$ is equal to $\|g\|_{\mathcal{L}^1(\mathbb{R}_+))}$ as in the previous evaluation. Hence

$$\begin{aligned}
|A_{114}(\sigma)| &\leq \sum_{k=0}^{m} \binom{m}{k} \sup_{1+\delta<t<+\infty} L_\sigma(t) \|g\|_{\mathcal{L}^1(\mathbb{R}_+)} + |g(z)| \int_{1+\delta}^{+\infty} L_\sigma(t) \frac{dt}{t} \\
&\leq 2^m \sup_{1+\delta<t<+\infty} L_\sigma(t) \|g\|_{\mathcal{L}^1(\mathbb{R}_+)} + |g(z)| \int_{1+\delta}^{+\infty} L_\sigma(t) \frac{dt}{t}
\end{aligned}$$

by the identity in equation (4.2.3), $\sum_{k=0}^{m} \binom{m}{k}$ is equal to 2^m.

By condition (c), we see that $\lim_{\sigma \to \sigma_0} |A_{114}(\sigma)| = 0$.

Since

$$|A_2(\sigma)| = \left|\int_0^{+\infty} K_\sigma\left(t, \sum_{k=1}^{m} \binom{m}{k} (-1)^{k-1} g(z)\right)\frac{dt}{t} - g(z)\right|,$$
$$= \left|\int_0^{+\infty} K_\sigma\,(t, g(z))\frac{dt}{t} - g(z)\right|$$

by the identity in equation (4.2.2) $\sum_{k=1}^{m}\binom{m}{k}(-1)^{k-1}$ is equal to 1, by condition (d), we see that $\lim_{\sigma\to\sigma_0} |A_2(\sigma)| = 0$. Thus, the proof is completed for $p = 1$.

The proof for the case $p > 1$ is as follows:

$$\left|T_\sigma^{[m]}(g;z) - g(z)\right| = \left|\int_0^{+\infty} K_\sigma\left(t, \sum_{k=1}^{m} \binom{m}{k} (-1)^{k-1}\, g(zt^k)\right)\frac{dt}{t} - g(z)\right|.$$

Adding and subtracting the expression $\int_0^{+\infty} K_\sigma\left(t, \sum_{k=1}^{m}\binom{m}{k}(-1)^{k-1}\, g(z)\right)\frac{dt}{t}$ to the expression inside the absolute value on the right-hand side above, we have

$$\left|\int_0^{+\infty} K_\sigma\left(t, \sum_{k=1}^{m} \binom{m}{k} (-1)^{k-1}\, g(zt^k)\right)\frac{dt}{t} - g(z) \mp \int_0^{+\infty} K_\sigma\left(t, \sum_{k=1}^{m} \binom{m}{k} (-1)^{k-1}\, g(z)\right)\frac{dt}{t}\right|.$$

Using triangle inequality, we get

$$\left|T_\sigma^{[m]}(g;z) - g(z)\right|$$

$$\leq \left| \int_0^{+\infty} K_\sigma\left(t, \sum_{k=1}^{m} \binom{m}{k} (-1)^{k-1}\, g(zt^k)\right) \frac{dt}{t} - \int_0^{+\infty} K_\sigma\left(t, \sum_{k=1}^{m} \binom{m}{k} (-1)^{k-1}\, g(z)\right) \frac{dt}{t} \right|$$

$$+ \left| \int_0^{+\infty} K_\sigma\left(t, \sum_{k=1}^{m} \binom{m}{k} (-1)^{k-1}\, g(z)\right) \frac{dt}{t} - g(z) \right|.$$

Using the inequality $(|a_1| + |a_2|)^p \leq 2^p(|a_1|^p + |a_2|^p)$ with $a_1, a_2 \in \mathbb{R}$ and $p \geq 1$, we have

$$\left| T_\sigma^{[m]}(g; z) - g(z) \right|^p$$

$$\leq 2^p \left| \int_0^{+\infty} K_\sigma\left(t, \sum_{k=1}^{m} \binom{m}{k} (-1)^{k-1}\, g(zt^k)\right) \frac{dt}{t} - \int_0^{+\infty} K_\sigma\left(t, \sum_{k=1}^{m} \binom{m}{k} (-1)^{k-1}\, g(z)\right) \frac{dt}{t} \right|^p$$

$$+ 2^p \left| \int_0^{+\infty} K_\sigma\left(t, \sum_{k=1}^{m} \binom{m}{k} (-1)^{k-1}\, g(z)\right) \frac{dt}{t} - g(z) \right|^p$$

$$=: 2^p\big(B_1(\sigma) + B_2(\sigma)\big).$$

First, we show that $B_1(\sigma) \to 0$ as $\sigma \to \sigma_0$.

Using the Lipschitz condition, we write

$$B_1(\sigma) \leq \left(\int_0^{+\infty} L_\sigma(t) \left| \sum_{k=1}^{m} \binom{m}{k} (-1)^{k-1}\, g(zt^k) - g(z) \right| \frac{dt}{t} \right)^p$$

$$= \left(\int_0^{+\infty} L_\sigma(t) \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{k-1}\, g(zt^k) \right| \frac{dt}{t} \right)^p$$

and applying Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p, q < +\infty$, we have

$$B_1(\sigma) \le \left(\int_0^{+\infty} L_\sigma(t) \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{k-1}\, g(zt^k) \right|^p \frac{dt}{t} \right) \left(\int_0^{+\infty} L_\sigma(t) \frac{dt}{t} \right)^{\frac{p}{q}}.$$

By condition (g), we get

$$B_1(\sigma) \le (M)^{\frac{p}{q}} \left(\int_0^{+\infty} L_\sigma(t) \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{k-1}\, g(zt^k) \right|^p \frac{dt}{t} \right)$$

$$= (M)^{\frac{p}{q}} B_{11}(\sigma).$$

Let $\delta \in (0, 1)$. We can split the integral $B_{11}(\sigma)$ into four terms as follows:

$$\begin{aligned} B_{11}(\sigma) = & \int_0^{1-\delta} L_\sigma(t) \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k}\, g(zt^k) \right|^p \frac{dt}{t} \\ & + \int_{1-\delta}^{1} L_\sigma(t) \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k}\, g(zt^k) \right|^p \frac{dt}{t} \\ & + \int_{1}^{1+\delta} L_\sigma(t) \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k}\, g(zt^k) \right|^p \frac{dt}{t} \\ & + \int_{1+\delta}^{+\infty} L_\sigma(t) \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k}\, g(zt^k) \right|^p \frac{dt}{t} \\ & =: B_{111}(\sigma) + B_{112}(\sigma) + B_{113}(\sigma) + B_{114}(\sigma). \end{aligned}$$

In view of equations (4.2.4) and (4.2.5) and using the definition of limit, we can write that for a given $\epsilon > 0$, there exists $\delta_{**} > 0$ such that:

$$\left|\left(\frac{1}{(\ln(1-h))^{1+\alpha}} \int_{1-h}^{1} \left|\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k}\, g(zt^k)\right|^p \frac{dt}{t}\right) - 0\right| \leq \epsilon \tag{4.2.10}$$

and

$$\left|\left(\frac{1}{(\ln(1+h))^{1+\alpha}} \int_{1}^{1+h} \left|\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k}\, g(zt^k)\right|^p \frac{dt}{t}\right) - 0\right| \leq \epsilon \tag{4.2.11}$$

hold provided that $0 < h \leq \delta < \delta_{**}$.

The relations (4.2.10) and (4.2.11) can be written respectively as follows:

$$\int_{1-h}^{1} \left|\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k}\, g(zt^k)\right|^p \frac{dt}{t} \leq \epsilon |\ln(1-h)|^{1+\alpha} \tag{4.2.12}$$

and

$$\int_{1}^{1+h} \left|\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k}\, g(zt^k)\right|^p \frac{dt}{t} \leq \epsilon |\ln(1+h)|^{1+\alpha} \tag{4.2.13}$$

for $\epsilon > 0$ and $0 < h \leq \delta < \delta_{**}$.

Now, we compute $B_{112}(\sigma)$ as follows:

In view of relation (4.2.12) and the auxiliary function:

$$F_1(t) := \int_{t}^{1} \left|\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k}\, g(zu^k)\right|^p \frac{du}{u},$$

we see that $|F_1(t)| \le \epsilon |\ln(t)|^{1+\alpha}$ for $t \in [1-\delta, 1)$. Using $F_1(t)$, we obtain the following equality:

$$|B_{112}(\sigma)| = \left| \int_{1-\delta}^{1} L_\sigma(t) \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} g(zt^k) \right|^p \frac{dt}{t} \right|$$
$$= \left| \int_{1-\delta}^{1} L_\sigma(t) dF_1(t) \right|.$$

Using integration by parts, we have

$$|B_{112}(\sigma)| = \left| \int_{1-\delta}^{1} L_\sigma(t) dF_1(t) \right|$$
$$= \left| [F_1(t) L_\sigma(t)]_{1-\delta}^{1} - \int_{1-\delta}^{1} F_1(t) dL_\sigma(t) \right|$$
$$= \left| [0 - F_1(1-\delta) L_\sigma(1-\delta)] - \int_{1-\delta}^{1} F_1(t)\, dL_\sigma(t) \right|$$
$$\le \epsilon |\ln(1-\delta)|^{1+\alpha} L_\sigma(1-\delta) + \epsilon \left| \int_{1-\delta}^{1} |\ln(t)|^{1+\alpha} dL_\sigma(t) \right|.$$

Since $L_\sigma(t)$ is the non-decreasing function of t on $[1-\delta, 1)$, $dL_\sigma(t) \ge 0$ for each fixed σ. Therefore, we have

$$|B_{112}(\sigma)| \le \left| \epsilon |\ln(1-\delta)|^{1+\alpha} L_\sigma(1-\delta) + \epsilon \int_{1-\delta}^{1} |\ln(t)|^{1+\alpha} dL_\sigma(t) \right|.$$

We will integrate this part: $\int_{1-\delta}^{1} |\ln(t)|^{1+\alpha}\, dL_\sigma(t)$.

Using integration by parts method, that is, letting $u = |\ln(t)|^{1+\alpha}$ with $du = (1 + \alpha)|\ln(t)|^{\alpha} \frac{dt}{t}$ and $dv = dL_\sigma(t)$ with $v = L_\sigma(t)$, we have

$$\int_{1-\delta}^{1} |\ln(t)|^{1+\alpha}\, dL_\sigma(t)$$

$$= [|\ln(t)|^{1+\alpha}\, L_\sigma(t)]_{1-\delta}^{1} - (1+\alpha) \int_{1-\delta}^{1} L_\sigma(t)|\ln(t)|^{\alpha} \frac{dt}{t}$$

$$= [0 - |\ln(1-\delta)|^{1+\alpha}\, L_\sigma(1-\delta)] - (1+\alpha) \int_{1-\delta}^{1} L_\sigma(t)|\ln(t)|^{\alpha} \frac{dt}{t}.$$

Collecting all the terms, we get

$$|B_{112}(\sigma)| \leq \left| \epsilon|\ln(1-\delta)|^{1+\alpha} L_\sigma(1-\delta) - |\ln(1-\delta)|^{1+\alpha}\, L_\sigma(1-\delta) \right.$$
$$\left. - \epsilon(1+\alpha) \int_{1-\delta}^{1} L_\sigma(t)|\ln(t)|^{\alpha} \frac{dt}{t} \right|.$$

Thus

$$|B_{112}(\sigma)| \leq \epsilon(1+\alpha) \int_{1-\delta}^{1} L_\sigma(t)|\ln(t)|^{\alpha} \frac{dt}{t}.$$

In view of relation (4.2.13) and the auxiliary function:

$$H_1(t) = \int_1^t \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k}\, g(zu^k) \right|^p \frac{du}{u},$$

we see that $|H_1(t)| \leq \epsilon|\ln(t)|^{1+\alpha}$ for $t \in (1, 1+\delta]$. Using $H_1(t)$ we obtain the following equality:

$$|B_{113}(\sigma)| = \left| \int_1^{1+\delta} L_\sigma(t) \left| \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k}\, g(zt^k) \right|^p \frac{dt}{t} \right|$$

$$= \left| \int_1^{1+\delta} L_\sigma(t) dH_1(t) \right|.$$

Using integration by parts method, we have

$$|B_{113}(\sigma)| = \left| \int_1^{1+\delta} L_\sigma(t) dH_1(t) \right|$$

$$= \left| [H_1(t)L_\sigma(t)]_1^{1+\delta} - \int_1^{1+\delta} H_1(t) dL_\sigma(t) \right|$$

$$= \left| [H_1(1+\delta)L_\sigma(1+\delta) - 0] - \int_1^{1+\delta} H_1(t)\, dL_\sigma(t) \right|$$

$$\leq \epsilon |\ln(1+\delta)|^{1+\alpha} L_\sigma(1+\delta) + \epsilon \left| \int_1^{1+\delta} |\ln(t)|^{1+\alpha} dL_\sigma(t) \right|.$$

Since $L_\sigma(t)$ is the non-increasing function of t on $(1, 1+\delta]$, $dL_\sigma(t) \leq 0$ for each fixed σ. Therefore

$$|B_{113}(\sigma)| \leq \left| \epsilon |\ln(1+\delta)|^{1+\alpha} L_\sigma(1+\delta) - \epsilon \int_1^{1+\delta} |\ln(t)|^{1+\alpha} dL_\sigma(t) \right|.$$

We will integrate this part: $\int_1^{1+\delta} |\ln(t)|^{1+\alpha}\, dL_\sigma(t)$.

Using integration by parts method, that is, letting $u = |\ln(t)|^{1+\alpha}$ with $du = (1+\alpha)|\ln(t)|^\alpha \frac{dt}{t}$ and $dv = dL_\sigma(t)$ with $v = L_\sigma(t)$, we have

$$\int_1^{1+\delta} |\ln(t)|^{1+\alpha}\, dL_\sigma(t)$$

$$= [|\ln(t)|^{1+\alpha} L_\sigma(t)]_1^{1+\delta} - (1+\alpha)\int_1^{1+\delta} L_\sigma(t)|\ln(t)|^\alpha \frac{dt}{t}$$

$$= [|\ln(1+\delta)|^{1+\alpha} L_\sigma(1+\delta) - 0] - (1+\alpha)\int_1^{1+\delta} L_\sigma(t)|\ln(t)|^\alpha \frac{dt}{t}.$$

Collecting all the terms, we get

$$|B_{113}(\sigma)| \leq \epsilon(1+\alpha)\int_1^{1+\delta} L_\sigma(t)|\ln(t)|^\alpha \frac{dt}{t}.$$

Hence,

$$(|B_{112}(\sigma)| + |B_{113}(\sigma)|) \leq \epsilon(1+\alpha)\left\{\int_{1-\delta}^{1} L_\sigma(t)|\ln(t)|^\alpha \frac{dt}{t} + \int_1^{1+\delta} L_\sigma(t)|\ln(t)|^\alpha \frac{dt}{t}\right\}$$

$$= \epsilon(1+\alpha)\int_{1-\delta}^{1+\delta} L_\sigma(t)|\ln(t)|^\alpha \frac{dt}{t}.$$

Since $\int_{1-\delta}^{1+\delta} L_\sigma(t)|\ln(t)|^\alpha \frac{dt}{t}$ is bounded for each fixed $\delta > 0$ and $0 \leq \alpha < \eta$ by condition (f) the result follows, that is,

$$\lim_{\sigma\to\sigma_0} (|B_{112}(\sigma)| + |B_{113}(\sigma)|) = 0.$$

Now, we will show that $B_{111}(\sigma) \to 0$ as $\sigma \to \sigma_0$.

Since

$$|B_{111}(\sigma)| = \left|\int_0^{1-\delta} L_\sigma(t)\left|\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k} g(zt^k)\right|^P \frac{dt}{t}\right|$$

$$= \left|\int_0^{1-\delta} L_\sigma(t)\left|\sum_{k=1}^{m}\binom{m}{k}(-1)^{m-k} g(zt^k) - g(z)\right|^P \frac{dt}{t}\right|,$$

using the inequality $(|a_1|+|a_2|)^p \le 2^p(|a_1|^p+|a_2|^p)$ with $a_1, a_2 \in \mathbb{R}$ and $p \ge 1$, we have

$$|B_{111}(\sigma)|$$
$$\le 2^p\left(\sum_{k=1}^{m}\binom{m}{k}\right)^p \sup_{0<t<1-\delta} L_\sigma(t)\int_0^{+\infty}|g(zt^k)|^p\frac{dt}{t} + 2^p|g(z)|^p\int_0^{1-\delta}L_\sigma(t)\frac{dt}{t}$$
$$\le 2^p\left(\sum_{k=0}^{m}\binom{m}{k}\right)^p \sup_{0<t<1-\delta} L_\sigma(t)\left(\|g\|_{\mathcal{L}^p(\mathbb{R}_+)}\right)^p + 2^p|g(z)|^p\int_0^{1-\delta}L_\sigma(t)\frac{dt}{t}$$
$$\le 2^p(2^m)^p \sup_{0<t<1-\delta} L_\sigma(t)\left(\|g\|_{\mathcal{L}^p(\mathbb{R}_+)}\right)^p + 2^p|g(z)|^p\int_0^{1-\delta}L_\sigma(t)\frac{dt}{t}$$
$$= 2^{p+mp} \sup_{0<t<1-\delta} L_\sigma(t)\left(\|g\|_{\mathcal{L}^p(\mathbb{R}_+)}\right)^p + 2^p|g(z)|^p\int_0^{1-\delta}L_\sigma(t)\frac{dt}{t}.$$

Using condition (c), we see that $\lim_{\sigma\to\sigma_0}|B_{111}(\sigma)| = 0$.

Similarly, we will show that $B_{114}(\sigma) \to 0$ as $\sigma \to \sigma_0$.

Since

$$|B_{114}(\sigma)| = \left|\int_{1+\delta}^{+\infty} L_\sigma(t)\left|\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}g(zt^k)\right|^P\frac{dt}{t}\right|$$
$$= \left|\int_{1+\delta}^{+\infty} L_\sigma(t)\left|\sum_{k=1}^{m}\binom{m}{k}(-1)^{m-k}g(zt^k)-g(z)\right|^P\frac{dt}{t}\right|,$$

using the inequality $(|a_1|+|a_2|)^p \le 2^p(|a_1|^p+|a_2|^p)$ with $a_1, a_2 \in \mathbb{R}$ and $p \ge 1$, we have

$$|B_{114}(\sigma)| \le 2^p\left(\sum_{k=0}^{m}\binom{m}{k}\right)^p \sup_{1+\delta<t<+\infty} L_\sigma(t)\int_0^{+\infty}|g(zt^k)|^p\frac{dt}{t} + 2^p|g(z)|^p\int_{1+\delta}^{+\infty}L_\sigma(t)\frac{dt}{t}$$

$$= 2^p \left(\sum_{k=0}^{m} \binom{m}{k} \right)^p \sup_{1+\delta<t<+\infty} L_\sigma(t) \left(\|g\|_{\mathcal{L}^p(\mathbb{R}_+)} \right)^p + 2^p |g(z)|^p \int_{1+\delta}^{+\infty} L_\sigma(t) \frac{dt}{t}$$

$$= 2^p (2^m)^p \sup_{1+\delta<t<+\infty} L_\sigma(t) \left(\|g\|_{\mathcal{L}^p(\mathbb{R}_+)} \right)^p + 2^p |g(z)|^p \int_{1+\delta}^{+\infty} L_\sigma(t) \frac{dt}{t},$$

$$= 2^{p+mp} \sup_{1+\delta<t<+\infty} L_\sigma(t) \left(\|g\|_{\mathcal{L}^p(\mathbb{R}_+)} \right)^p + 2^p |g(z)|^p \int_{1+\delta}^{+\infty} L_\sigma(t) \frac{dt}{t}.$$

Using condition (c), we see that $\lim_{\sigma \to \sigma_0} |B_{114}(\sigma)| = 0$.

Lastly, we will show that $|B_2(\sigma)| \to 0$ as $\sigma \to \sigma_0$.
Since

$$|B_2(\sigma)| = \left| \int_0^{+\infty} K_\sigma \left(t, \sum_{k=1}^{m} \binom{m}{k} (-1)^{k-1} g(z) \right) \frac{dt}{t} - g(z) \right|^p$$

$$= \left| \int_0^{+\infty} K_\sigma \left(t, g(z) \right) \frac{dt}{t} - g(z) \right|^p,$$

by condition (d), we see that $\lim\limits_{\sigma \to \sigma_0} |B_2(\sigma)| = 0$.

Thus, the proof is completed.

Now, we give some graphical examples.

Example 4.2.2. Let $n \in \mathbb{N}$. Bardaro and Mantellini (2006) considered the following respectively linear and nonlinear moment operators:

$$\mathrm{M}_n^*(g; z) = \int_0^{+\infty} K_n^* (tz^{-1}) g(t) \frac{dt}{t}, \tag{4.2.14}$$

where $K_n^* = \begin{cases} nt^n, & if \quad t \in (0,1) \\ 0, & if \quad t \in \mathbb{R}_+ \setminus (0,1) \end{cases}$

and

$$\mathrm{M}_n^{**}(h; z) = \int_0^{+\infty} K_n^* (tz^{-1}) G_n\big(h(t)\big) \frac{dt}{t}, \qquad (4.2.15)$$

where $G_n(v) = \frac{nv|v|}{n|v|+1}, v \in \mathbb{R}$.

The analysis of these operators can also be found in (Bardaro and Mantellini, 2006) with some examples. Bardaro et al. (2013) remarked that the theory for Mellin-type nonlinear m −singular integral operators is compatible with moment-type operators via using the operators defined in equation (4.2.1).

The Mellin-type $m -$ singular moment operators with respect to (4.2.14) and (4.2.15) may be written as:

$$\mathrm{M}_n^{*[m]}(g; z) = \int_0^{+\infty} \mathrm{K}_n^*(t) \left[\sum_{k=1}^{m} \binom{m}{k} (-1)^{k-1} g(zt^k) \right] \frac{dt}{t}, \qquad (4.2.16)$$

and

$$\mathrm{M}_n^{**[m]}(h; z) = \int_0^{+\infty} \mathrm{K}_n^* (t) \left[\sum_{k=1}^{m} \binom{m}{k} (-1)^{k-1} G_n\Big(h(zt^k)\Big) \right] \frac{dt}{t}. \qquad (4.2.17)$$

Now, we present Figures 4.1 and 4.2, which are generated by using computer algebra system Mathematica 12.2.

Let $g(t) = \sqrt{t}e^{-t}$ with $t \in \mathbb{R}_+$. We applied the operators defined in (4.2.16) to the function $g(t)$. In Figure 4.1, the thick and orange graph represents the case $m = 3$, and $n = 4$, thick and dashed graph represents the case $m = 1$ and $n = 4$, the thick and purple graph represents the case $m = 2$ and $n = 4$ and thick and black graph represents the original function.

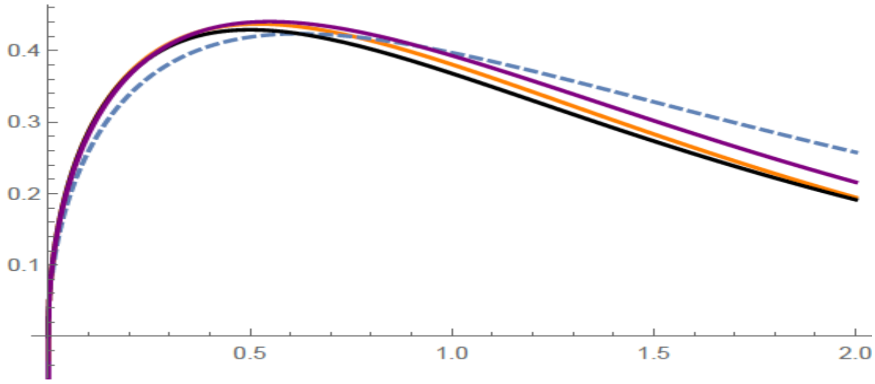

Figure 4.1. *Approximation by linear moment-type operators*

Let $h(t) = \begin{cases} t^2, & if \quad t \in (0,1), \\ 0, & if \ \ t \in \mathbb{R}_+ \backslash (0,1). \end{cases}$ We applied the operators defined in (4.2.17) to the function $h(t)$. In Figure 4.2, the thick and pink graph represents the case $m = 1$ and $n = 8$, the thick and dashed graph represents the case $m = 2$ and $n = 8$, the thick and black graph represents the case $m = 3$ and $n = 8$ and thick and dark blue graph represents the original function.

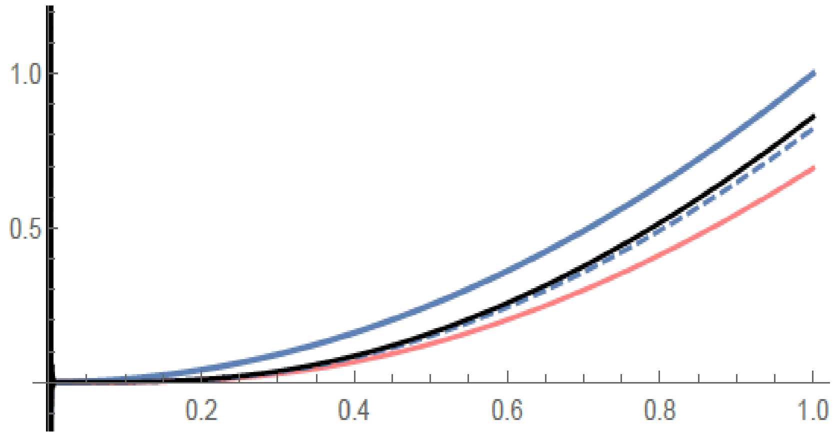

Figure 4.2. *Approximation by nonlinear moment-type operators*

PART 5

CONCLUSION

In this thesis, a pointwise convergence result (Theorem 4.2.1) concerning generalized Mellin $m-p-$ Lebesgue points of integrable functions is proved by using some theorems in (Bardaro et al., 2013) and (Mamedov, 1991). The result is supported with some graphical examples, which are generated by using computer algebra system Mathematica 12.2. The result with some additional generalizations will be presented in a conference, possibly in 2021, by D. Q. Haso as it is a joint work with G. Uysal.

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RESUME

Dilshad Qasim Hamza HASO moved to Duhok in order to pursue his high school studies there. He finished his primary education in Duhok in 2013. He started university study at Duhok University- College of Science- Department of Mathematics. He wrote a B.Sc. thesis entitled Complex Integrations which was part of the requirements to obtain a B.Sc. in Mathematics in 2017. He has taught mathematics as a volunteer in Duhok High School for a year. HASO speaks four languages, including Kurdish, Arabic, Turkish, and English. In 2019, he started his post-graduate studies at Karabük University.