EXAMINATION OF DIFFERENT SOLUTION APPROACHES FOR CERTAIN TYPES OF COMPLEX INTEGRALS

2022
MASTER THESIS
MATHEMATICS

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EXAMINATION OF DIFFERENT SOLUTION APPROACHES FOR CERTAIN TYPES OF COMPLEX INTEGRALS

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Master Thesis

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April 2022

I certify that in my opinion the thesis submitted by Ayhan Jalal KHALİD titled "EXAMINATION OF DIFFERENT SOLUTION APPROACHES FOR CERTAIN TYPES OF COMPLEX INTEGRALS" is fully adequate in scope and in quality as a thesis for the degree of Master of Science.

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ABSTRACT

M. Sc. Thesis

EXAMINATION OF DIFFERENT SOLUTION APPROACHES FOR CERTAIN TYPES OF COMPLEX INTEGRALS

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April 2022, Pages 33

In this thesis, three different solution approaches are investigated for complex integrals of a certain type. These approaches are analyzed and applied to a similar kind of integral as an example in various chapters. The first approach is the solution by the method of saddle point formula for the presented complex integral. The second approach is the deformation of the integration contour onto branch cuts and calculation of integral over the branch cut. The third technique is to introduce a smooth contour formulation to calculate these types of complex integral numerically. Moreover, only the saddle point method produces an analytical result for large parameter values. In addition, in the other approaches, the results are still in an integral form. Branch cut integration approach results with an infinite integral while smooth contour approach results with a finite integral. However, in both approaches the obtained integrals have become more suitable for parametric numerical studies. Numerical calculations for different integral parameters showed that all three approaches in a good aggreament at

higher frequencies as expected while saddle point formula seems far from being compatible with the other two methods at lower frequencies.

Key Words **:** Complex integration, Asymptotic evaluation, Branch cut.
Science Code : 20406

ÖZET

Yüksek Lisans Tezi

BELİRLİ KOMPLEKS İNTEGRAL TÜRLERİ İÇİN FARKLI ÇÖZÜM YAKLAŞIMLARININ İNCELENMESİ

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Nisan 2022, Sayfa 27

Bu tezde, belirli bir tipteki karmaşık integraller için üç farklı çözüm yaklaşımı incelenmiştir. Farklı bölümlerde bu yaklaşımlar analiz edilmiş ve örnek olarak benzer türdeki bir integrale uygulanmıştır. Birinci yaklaşım, bu karmaşık integral için semer noktası formülü yöntemiyle çözüm yöntemi, ikinci yaklaşım, integral konturunun kesim çizgileri üzerine deformasyonu ve bu kesim çizgileri üzerinden integralin hesaplanmasıdır. Üçüncü yaklaşım ise, bu tür karmaşık integralleri sayısal olarak hesaplamak için düzgün bir kontur formülasyonu sunmaktır. Yalnızca semer noktası yöntemi büyük parametre değerleri için analitik sonuçlar üretirken, diğer yaklaşımlarda sonuçlar hala integral formdadır. Kesim çizgisi integrasyon yaklaşımı sonsuz bir integral ile sonuçlanırken, düzgün kontur yaklaşımı sonlu bir integral ile sonuçlanmıştır. Ancak, her iki yaklaşımda da elde edilen integraller parametrik sayısal çalışmalar için daha uygun hale gelmiştir. Farklı integral parametreleri için yapılan sayısal hesaplamalar, her üç yaklaşımın da beklendiği gibi yüksek frekanslarda iyi bir

uyum içinde olduğunu gösterirken, eyer noktası formülü daha düşük frekanslarda diğer iki yöntemle uyumlu olmaktan uzak görünmektedir.

Anahtar Kelimeler : Kompleks entegrasyon, Asimptotik değerlendirme, Dal kesimi.
Bilim Kodu : 20406

ACKNOWLEDGMENT

I’m grateful to every single person who lent a hand to guide me to make my dissertation possible. Especially my warmest praises to my supervisor Prof. Dr. Ahmet DEMİR, for all his guidance.

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SYMBOLS AND ABBREVIATIONS INDEX

SYMBOLS

$H_n^{(2)}$: Hankel function of the second kind of order n

$J_v(z)$: Bessel function of the first kind of order v

$I(\alpha)$: Analytic function

Zs : Saddle point

$H[\quad]$: Heaviside step function

ABBREVIATIONS

C : Contour in the complex z- plane

SDP : Saddle point technique

BC : Branch cut

SC : Smooth Contour

Res : Residue

L : Contour in the complex u- plane

LHS : Left-hand side

RHS : Right-hand side

CHAPTER 1

INTRODUCTION

The integration of complex functions plays a significant role in various areas of science and engineering. Complex integration is an intuitive extension of real integration. Many problems are solved and expressed as an integral formula in mathematics and physics. Mostly the problems are contingent on a parameter; as the parameter is closed to infinity, the more intrigued the solution's nature and behavior become. Finding an asymptotic expansion in the parameter is the most widely used way of evaluating the behavioral nature of these integral representations. We usually encounter the following form of the general integral in the analytical solutions of math-physics problems [1].

$$I(\Omega) = \int_C f(z)e^{\Omega q(z)}dz, \tag{1.1}$$

where functions *f* and *q* are analytical functions of the complex variable z, we assume the large parameter **Ω** to be positive [2] where C is a contour in the Complex plane. Moreover, many integrals transform exhibit this form such as the Laplace transform and the Fourier transform.

Wave radiation or wave diffraction problems in open regions generally result in a solution in the form of a given integral (1.1). The presented form of integral equations is complicated and cannot be expressed as a solution in terms of simple functions in closed form. However, for the large values of parameter **Ω**, it is possible to evaluate such integrals asymptotically [3].

In addition, the problem of radiation and propagation of waves in waveguides, scattering, diffraction of waves from obstacles, and propagation of waves through layered mediums have received a great deal of importance and attention. Numerous studies have been published in this era [4,5,6,7]. The common point of these studies is that the solution is always obtained by a parameter-dependent contour integral. On the other hand, the integral is calculated asymptotically for large values of this integral parameter, and the high-frequency solution is obtained analytically and numerically.

Sommerfeld [8] was the first to look for a solution for electromagnetic wave propagation in layered environments. On the other hand, Vajnshtejn [27] was the first to solve the radiation emanating from the output of a semi-infinite circular channel for electromagnetic waves and later adapted it to sound waves. The results have been generalized and explored by many acousticians [1].

One of the methods of solving boundary-value problems involving wave radiation or wave scattering in electromagnetics or acoustics is the Wiener-Hopf technique, which is very well known and has been successfully applied to the various problem for many years [3]. In such problems, one is expected one to solve Helmholtz equation under some boundary conditions depending on problem geometry and material properties of a boundary. In the process of Wiener-Hopf technique solving these kinds of problems, we first take Fourier transform of Helmholtz equation defined in suitable coordinate system with the problem geometry. For example, in the cylindrical coordinate system without angular dependency (it is the case where the angular symmetry exists), the Helmholtz equation and its Fourier transform are defined as :

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{\partial^2}{\partial z^2}+k^2\right]u(r,z)=0 \quad , \tag{1.2a}$$

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+(k^2-\alpha^2)\right]F(r,\alpha)=0 \ , \tag{1.2b}$$

where $u(r,z)$ is unknown scalar field having Fourier transform $F(r,\alpha)$ under transform parameter α ,

$$F(r,\alpha)=\int_{-\infty}^{\infty}u(r,z)e^{i\alpha z}dz \ . \tag{1.3}$$

The solution of (1.2b) can be expressed in terms of Bessel functions $H_0^{(1,2)}, J_0, Y_0$ depending on the interval of the definition of r. In the next steps of Wiener-Hopf technique, we encounter a functional equation after applying transformed boundary and continuity relations. This equation usually has the following form

$$M(\alpha)F_+(\alpha) + G_-(\alpha) = N(\alpha) \tag{1.4}$$

holds in the strip $\tau_- < Im(\alpha) < \tau_+$, $-\infty < Re(\alpha) < \infty$ of the complex α- plane.

$F_+(\alpha)$ is an analytic function in the half plane $Im(\alpha) > \tau_-$ and $G_-(\alpha)$ is an analytic function in the half plane $Im(\alpha) < \tau_+$. $M(\alpha), N(\alpha)$ are analytic functions in the strip defined for (1.4). To solve functional equation (1.4) for $F_+(\alpha), G_-(\alpha)$, one must factorize $M(\alpha)$ first as $M(\alpha) = \frac{M_+}{M_-}$ and then decompose $N(\alpha)M_-(\alpha)$ as $N(\alpha)M_-(\alpha) = N_+(\alpha) + N_-(\alpha)$, where M_+, N_+ are analytic functions in the half plane $Im(\alpha) > \tau_-$ while M_-, N_- are analytic functions in the half plane $Im(\alpha) < \tau_+$. Factorization and decomposition are very important steps in the application of W-H technique. In both steps, M_+, N_+, M_-, N_- are split functions can be obtained via contour integrals under some conditions.

In some problems, we can come up with integrals of type (1.1) when applying factorization or decomposition procedures of W-H technique. An example of this can be seen in [4]. In this study, Buyukaksoy and Polat studied the problem of diffraction of acoustic waves emanating from a ring source by a semi-infinite cylindrical pipe of certain wall thickness having different internal, external and end surface impedances. They solved the related boundary-value problem via Wiener-Hopf technique in conjunction with the Mode-Matching method. In the decomposition step of Wiener-Hopf technique, they encountered an integral of type (1.1) and evaluated that integral by means of the steepest-descent method. Following this study, Tiryakioglu and Demir analyzed diffraction of sound waves emanating from a ring source by a rigid cylindrical pipe with external impedance surface solving the related boundary-value problem by Wiener-Hopf technique [9]. They similarly encountered with an integral of type (1.1) in the decomposition step of W-H technique and solved this integral deforming the contour onto the branch cut.

In this study, we will focus on examining different evaluation approaches in details such as saddle-point method and contour deformations of integrals of type (1.1) encountered during the solution process of boundary-value problems especially W-H technique is applied. For this purpose, without losing parametric dependency of the integral function, a simplified version of the integral in [9] is considered.

$$I(\alpha) = \frac{1}{2\pi i}\int_{L_+} \frac{H_0^{(2)}(Kb)e^{-iuc}}{KH_1^{(2)}(Ka)(u-\alpha)} du \tag{1.5}$$

Suppose that $R_0 > 0, and\ 0 < \theta_0 < \pi$, with

$$b - a = R_0 sin\theta_0 \quad , \quad c = R_0 cos\theta_0 \quad , \tag{1.6}$$

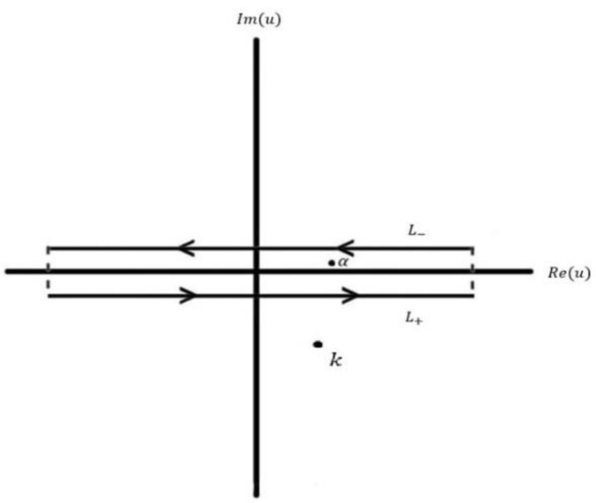

Figure 1.1. Integration contour of $I(\alpha)$ in the complex plane.

where L_+ is a suitable integration contour along the real axis in the complex plane. $H_n^{(2)} = J_n - iY_n$ is the Hankel function of the second kind, and nth-order K is the square root function which is defined as:

$$K = \sqrt{k^2 - u^2}, \tag{1.7}$$

where k is the free-space wave number which is temporarily considered to have a negative imaginary part to obtain a unique solution [4].

In this research, five chapters have been presented. The first chapter introduces the problem. The second chapter deals with the method for the solution by the technique of the saddle point formula for the proposed type of complex integral. However, in the third chapter, the deformation of the integration contour onto Branch cuts is presented. In the same chapter, the integral calculation over the Branch cut has been obtained. However, this chapter involves basic definitions and theorems stated to understand the integration of analytic functions over given contours.

On the other hand, in the fourth chapter, a smooth contour formulation is defined to calculate the mentioned kinds of complex integrals numerically. Following chapter is devoted to numerical studies. In this chapter, formulas obtained from different solution approaches for $I(\alpha)$ in Chapters 2,3,4 have been evaluated numerically on computer for different values of parameters a,b,c and k. Finally, the last chapter is the conclusion of the work. A list of used references is presented at the end of the research.

CHAPTER 2

SADDLE POINT TECHNIQUE

In mathematics, the steepest descent or saddle-point method are extensions of Laplace's method for approximating an integral, where one deforms a contour integral in the complex plane to pass near a stationary point (saddle point), in roughly the direction of steepest descent or stationary phase.

The saddle point method is very useful for obtaining asymptotic estimates on the coefficients of an analytic function via the evaluation of contour integrals. Its general approach is to use a contour that crosses a so-called "saddle point", where the modulus of the integrand is maximized, and then locally estimate the integral near this point.

The basic idea of the method of steepest descent (or sometimes referred to as the saddle-point method) is that we apply Cauchy's theorem to deform the contour C to contours coinciding with the path of steepest descent.

The saddle-point approximation is used with integrals in the complex plane. In contrast, Laplace's method is used with real integrals. Radiation and diffraction fields in open regions (with infinite cross section) are usually expressed by integral representations that cannot be evaluated in closed form. In many applications, the integrands contain a large parameter, gamma, in which the integrals may be approximated. While such an evaluation can be treated for rather general functional dependences of the integrand on gamma, it will suffice within the present context to consider the integrand of the proposed type.

Regarding the real integrals, Laplace's approach is used in open regions (with infinite cross section). House radiation and diffraction fields that cannot be evaluated in the form are expressed by integral representations of an asymptotic saddle point. However, the integrands contain a large parameter that can be approximated integrally [2]. Deforming the integration line C in equation(2.1) onto the steepest-descent path (SDP) through the saddle point z_s [4], we obtain

$$I(\Omega) = \int_{SDP} f(z)e^{\Omega q(z)}dz \quad , \tag{2.1}$$

where the function $f(z)$ has no singularities near an isolated first-order saddle point z_s of $q(z)$, and $q'(z_s) = 0$, $q''(z_s) \neq 0$. Asymptotic approximation of $I(\Omega)$ is given by

$$I(\Omega) \approx \sqrt{\frac{-2\pi}{\Omega q''(z_s)}}\, f(z_s)e^{\Omega q(z_s)} \ , \quad \Omega \to \infty \quad . \tag{2.2}$$

Along the SDP, when $q(z) = i\hat{q}z$ with $\hat{q}$ denoting a real function of z and z_s is real, equation(2.2) may be written as follows [2]:

$$I(\Omega) = \int_{SDP} f(z)e^{i\Omega\hat{q}(z)}dz \sim \sqrt{\frac{2\pi}{\Omega|\hat{q}''(z_s)|}}\, f(z_s)e^{i\Omega\hat{q}(z_s)\pm\frac{i\pi}{4}} \quad . \tag{2.3}$$

If $\hat{q}\,''(z_s) > 0$, then it approaches to $\frac{+i\pi}{4}$. Otherwise, $if\ \hat{q}\,''(z_s) < 0$, it approaches to $\frac{-i\pi}{4}$, which is provided Re(dz) increases along the SDP near z_s as shown in figure (2.1).

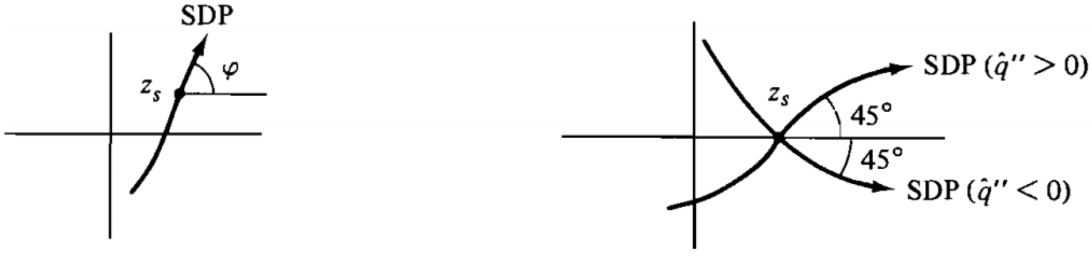

Figure 2.1. Integration paths in the z-plane [2].

2.1. ASYMPTOTIC EVALUATION BY SADDLE POINT TECHNIQUE AS THE METHOD OF SOLUTION

The steepest-Descent approach will be used to evaluate the integral equation presented in (1.5) by using the following substitution [4]:

$$u = k\cos\varepsilon \quad \Longrightarrow \quad du = -k\sin\varepsilon\, d\varepsilon$$

Putting the above assumption in (1.7), we obtain

$$K = \sqrt{k^2 - u^2} = \sqrt{k^2 - k^2\cos^2\varepsilon}$$
$$= k\sin\varepsilon \quad ,$$

Where

$$\sin\varepsilon = \sqrt{1 - \cos^2\varepsilon}$$

Putting the above assumptions on (1.5), we obtain

$$I(\alpha) = \frac{1}{2\pi i}\int_{\Gamma} \frac{H_0^{(2)}(kb\sin\varepsilon)e^{-ikc\cos\varepsilon}}{k\sin\varepsilon H_1^{(2)}(ka\sin\varepsilon)(k\cos\varepsilon - \alpha)}(-k\sin\varepsilon)d\varepsilon \quad , \tag{2.4}$$

where Γ as is shown in the following figure:

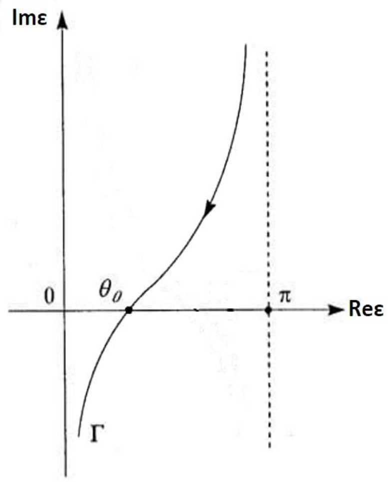

Figure 2.2. The complex ε-plane.

In the asymptotic expansion of Hankel's function for large arguments, it is possible to utilize the asymptotic formula for Hankel's function as follows:

$$H_0^{(2)}(kbsin\varepsilon) \approx \sqrt{\frac{2}{\pi kbsin\varepsilon}}\, e^{-ikbsin\varepsilon+\frac{i\pi}{4}} \qquad ,$$

$$H_1^{(2)}(kasin\varepsilon) \approx \sqrt{\frac{2}{\pi kasin\varepsilon}}\, e^{-ikasin\varepsilon+\frac{i3\pi}{4}} \qquad .$$

By making the substitutions, it yields

$$I(\alpha) = \frac{1}{2\pi i}\int_\Gamma \frac{\sqrt{\frac{2}{\pi kbsin\varepsilon}}\, e^{-ikbsin\varepsilon+\frac{i\pi}{4}} e^{-ikc\,cos\varepsilon}}{ksin\varepsilon\sqrt{\frac{2}{\pi kasin\varepsilon}}\, e^{-ikasin\varepsilon+\frac{i3\pi}{4}}(kcos\varepsilon-\alpha)}(-ksin\varepsilon\,)d\varepsilon\,.$$

By simplifying, it becomes

$$I(\alpha) = -\frac{1}{2\pi i}\sqrt{\frac{a}{b}}\int_\Gamma \frac{e^{-iksin\varepsilon(b-a)-\frac{i\pi}{2}}\, e^{-ikc\,cos\varepsilon}}{kcos\varepsilon-\alpha}\, d\varepsilon$$

Inserting the value of $e^{-\frac{i\pi}{2}} = -i$, we get

$$I(\alpha) = \frac{1}{2\pi}\sqrt{\frac{a}{b}}\int_{\Gamma} \frac{e^{-iksin\varepsilon(b-a)}\ e^{-ikc\,cos\varepsilon}}{kcos\varepsilon - \alpha}\ d\varepsilon\ . \tag{2.5}$$

Putting assumption (1.6) in (2.5), we obtain

$$I(\alpha) = \frac{1}{2\pi}\sqrt{\frac{a}{b}}\int_{\Gamma} \frac{e^{-ikR_0cos\,(\varepsilon-\theta_0)}}{kcos\varepsilon - \alpha}\ d\varepsilon\ \ .$$

Now, we will evaluate I_{res}.

$$I_{res} = \frac{H_0^{(2)}(Kb)}{KH_1^{(2)}(Ka)} e^{-iuc}\ \ H\left[arc\ cos\ \frac{\alpha}{k} - \theta_0\right], \tag{2.6}$$

since we have $kcos\varepsilon - \alpha$ in the denominator

$$kcos\varepsilon - \alpha = 0 \qquad \to cos\varepsilon = \frac{\alpha}{k} \quad \to\ \varepsilon = arc\ cos\ \left(\frac{\alpha}{k}\right).$$

Using (2.4) and (2.5), and applying saddle point formula, we have that

$$I(\alpha) = \frac{1}{2\pi}\sqrt{\frac{a}{b}}\int_{SDP} \frac{e^{-ikR_0cos\,(\varepsilon-\theta_0)}}{kcos\varepsilon - \alpha} + I_{res} \tag{2.7}$$

In equation (2.7), we obtain that

$$f(\alpha) = \frac{1}{2\pi}\sqrt{\frac{a}{b}}\ \frac{1}{kcos\varepsilon-\alpha}\quad ,$$

$$e^{i\Omega q(\varepsilon)}d\varepsilon = e^{-ikR_0cos\,(\varepsilon-\theta_0)}\,d\varepsilon\ \ ,$$

$$\Omega = k\ \ ,$$

$$\hat{q}(\varepsilon) = -R_0 \, cos \ (\varepsilon - \theta_0) \rightarrow \ \hat{q}'(\varepsilon) = R_0 \, sin \ (\varepsilon - \theta_0) \ \rightarrow \varepsilon_s = \theta_0 \ ,$$

$$\hat{q}''(\varepsilon_s) = R_0 > 0 \ .$$

Then

$$I(\alpha) \approx \sqrt{\frac{2\pi}{kR_0}} \left(\frac{1}{2\pi} \sqrt{\frac{a}{b}} \ \frac{e^{ik(-R_0)+\frac{i\pi}{4}}}{(kcos\theta_0 - \alpha)} \right) + I_{res} \ ,$$

$$I(\alpha) \approx \ e^{\frac{i\pi}{4}} \sqrt{\frac{a}{2\pi b}} \ \frac{1}{(kcos\theta_0 - \alpha)} \ \frac{e^{-ikR_0}}{\sqrt{kR_0}} \ + I_{res} \quad . \tag{2.8}$$

From (2.6) and (2.8), we obtain

$$I(\alpha) \approx \ e^{\frac{i\pi}{4}} \sqrt{\frac{a}{2\pi b}} \ \frac{1}{(kcos\theta_0 - \alpha)} \ \frac{e^{-ikR_0}}{\sqrt{kR_0}}$$
$$+ \frac{H_0^{(2)}(Kb)}{KH_1^{(2)}(Ka)} e^{-iuc} \ H \left[arc \ cos \ \frac{\alpha}{k} - \theta_0 \right] \ . \tag{2.9}$$

CHAPTER 3

BRANCH CUT INTEGRATION

The point z_0 is known as a branch point for a complex function with its values in the multiples value$f(z)$. If the value of $f(z)$ does not turn back to its initial value in the shape of a closed curve, then around the point is traced (starting from any point on the curve). Hence f varies regularly as the path gets traced [10]. A branch cut is a curve (with ends possibly open, closed, or half-open) in the complex plane across which a multivalued analytic function is discontinuous. For convenience, branch cuts are often taken as lines or line segments. Branch cuts are called cut lines [11], slits [12], or Branch lines. For example, take into consideration the function of z^2 which maps every complex number z to a well-defined number z^2. It is inverse function is $\sqrt{z}$. On the other hand, it is maps the value $z = 1$ to $\sqrt{z} = \mp 1$. While a unique principal value can be chosen for functions of this kind, the principal square root is positive. The choice cannot be made continuously over the whole complex plane. Instead, lines of discontinuity must occur. The most widely used method for approaching along with those discontinuous is the adoption of the commonly named branch cuts approach.

In general, branch cuts are not unique but are instead picked by conventional wisdom to give simple analytic properties. Several functions express relatively simple branch cut structure, while branch cuts for other functions are exceedingly complex [12].

3.1. SOME DEFINITIONS AND THEOREMS

To begin with there are a couple of definitions and expressions that needs to be brought up to clarify the material in upcoming sections.

Definition 3.1. (Analytic functions)

A function $f(z)$ is analytic on the open set U if $f(z)$ is complex differentiable at each point of U and the complex derivative $f'(x)$ is continuous on U [13].

Definition 3.2. (Pole and simple pole)

If z_0 is an isolated singularity, and there exists a positive integer n such that

$$(z - z_0)^n f(z) = A \neq 0,$$

then $z = z_0$ is called a pole of order n. If $n = 1$, z_0 is called a simple pole [14].

Theorem 3.1. (Cauchy's theorem):

Suppose that f is an analytic function in a simply connected domain D. Then, for every simple closed contour C in D,

$$\int_C f(z)\,dz = 0$$

[15].

Theorem 3.2. (Cauchy's integral formula):

Let $f(z)$ be analytic inside and on a simple closed curve C, and let a be any point inside C in figure (3.1) [14].

Then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - a}\,dz \quad , \tag{3.1}$$

where C is traversed in the positive (counterclockwise) sense.

Also, the nth derivative of $f(z)$ at $z = a$ is given by

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz \quad n = 1,2,3,\ldots \tag{3.2}$$

The result (3.1) can be considered as a special case of (3.2) with $n = 0$ if we define $0! = 1$.

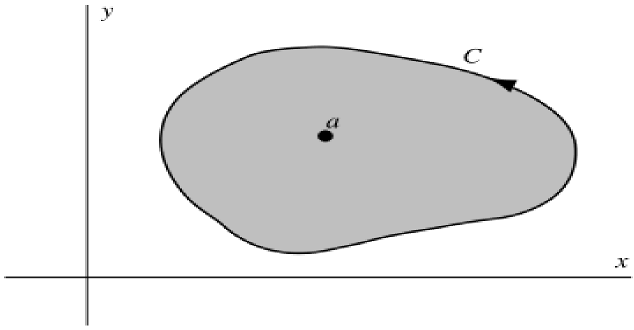

Figure 3.1. Simple closed curve C on complex z - plane.

Definition 3.3. (Residue)

Let z_0 be an isolated singular point of a function $f(z)$. Then the residue of $f(z)$ at z_0 denoted by

$$\operatorname*{Res}_{z=z_0} f(z)$$

[16].

It means the coefficients a_{-1} in the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n.$$

Theorem 3.3. (Residue theorem)

If a function $f(z)$ is analytic inside and on a piecewise smooth closed Jordan curve C except for isolated singular points $z_1, z_2, \ldots, z_n$ lying inside C, then

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{N} \operatorname*{Res}_{z=z_k} f(z) ,$$

[16].

3.2. EXAMPLE: INTEGRATION OF BRUNCH CUT

Example 3.1. Evaluate the following integral

$$\int_0^\infty \frac{1}{\sqrt{x}(x+1)}\, dx$$

Solution:

Firstly, observe that the real integral appears to be erroneous for a pair of causes. Bring attention to the infinite discontinuity at $x = 0$ and the infinite limit of integration. In addition, it is arguable, and the results show that the integrand behaves like $x^{\frac{-1}{2}}$ near the origin and like $x^{\frac{-3}{2}}$ as $x \to \infty$, that the integral converges, we will form the integral as follow:

$$\oint_C \frac{1}{z^{\frac{1}{2}}\ (z+1)}\, dz$$

As shown in Figure (3.2), C is the closed contour, and it forms of quadruple element: the parts of the circles are C_r and C_R. the parallel horizontal line segments are AB and ED, which run along opposite sides of the branch cut. The integrand of $f(z)$ is analytic, and the contour integral which is a line valued in and on the contour C, except for the pole, which is simple, $z = -1 = e^{\pi i}$ Therefore, we could formulate as

$$\oint_c \frac{1}{z^{\frac{1}{2}}\ (z+1)}\, dz = 2\pi i\, Res(f(z), -1)$$

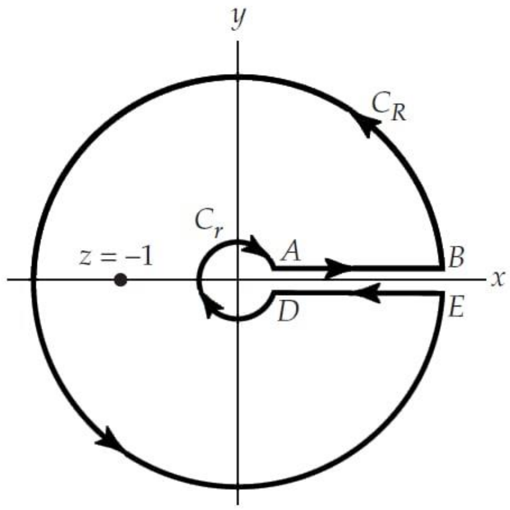

Figure 3.2. Contour for Example 3.2.

$$Or \quad \oint_{CR} + \oint_{ED} + \oint_{Cr} + \oint_{AB} = 2\pi i \; Res(f(z), -1)$$

Regardless of what is illustrated in Figure (3.2), it can be permissible to consider line segments AB and ED indeed stand on the positive real axis also more accurately.

AB corresponds with the higher side of the positive real axis for which $\theta = 0$ and ED corresponds with the minor side of the positive real axis for which $\theta = 2\pi$.

On AB, $z = xe^{0i}$, and on the ED, $z = xe^{(0+2\pi)i} = xe^{2\pi i}$, therefore, we obtain

$$\oint_{ED} = \int_R^r \frac{\left(xe^{2\pi i}\right)^{\frac{-1}{2}}}{xe^{2\pi i} + 1} \, (e^{2\pi i} dx) = -\int_R^r \frac{(x)^{\frac{-1}{2}}}{x+1} \, dx = \int_r^R \frac{(x)^{\frac{-1}{2}}}{x+1} \, dx$$

However, for AB, we get

$$\oint_{AB} = \int_r^R \frac{(xe^{0i})^{\frac{-1}{2}}}{xe^{0i} + 1} \, (e^{0i} dx) = \int_r^R \frac{(x)^{\frac{-1}{2}}}{x+1} \, dx \; .$$

Now, with $z = re^{i\theta}$ and $z = Re^{i\theta}$, on both C_r $and\, C_R$ subsequently. It can be illustrated by analysis congruent such

$$\oint_{Cr} \to 0 \quad , as \quad r \to \infty \qquad and \qquad \oint_{CR} \to 0 \;, as \quad R \to \infty \quad .$$

We obtain

$$\left[\oint_{CR} \quad + \oint_{ED} \quad + \oint_{cr} \quad + \oint_{AB} \quad = 2\pi i \; Res(f(z), -1) \right]$$

Hence, it is identical to

$$2 \int_0^{\infty} \frac{1}{\sqrt{x}(x+1)} \; dx = 2\pi i \; Res(f(z), -1)$$

Finally,

$$Res(f(z), -1) = z^{\frac{-1}{2}} |_{\; z = e^{\pi i}} = e^{\frac{-\pi i}{2}} = -i,$$

The result of the problem is

$$\int_0^{\infty} \frac{1}{\sqrt{x}(x+1)} \; dx = \pi$$

3.3. METHOD OF SOLUTION

This section calculates the integral defined in (1.5) by deforming the integration contour onto branch cut lines. We use the Cauchy theorem to the closed contour given in figure (3.3) during this process.

Applying the Cauchy theorem for the problem presented in (1.5), it seems $L_+, C_r, L_1, C_\epsilon, L_2$ is a closed curve with a negative direction.

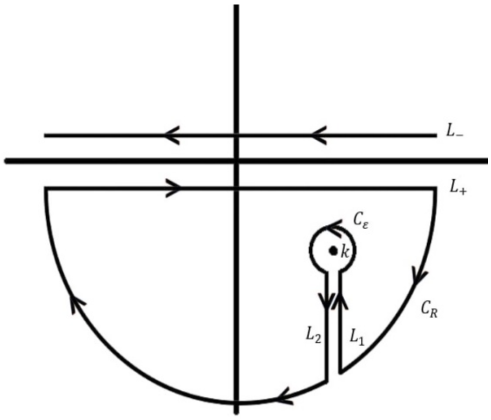

Figure 3.3. Integration contour around the brunch cut.

$-k+u=t\,e^{\frac{-i\pi}{2}}$ L_1: R H S of the branch cut $(0<t<\infty)$

$-k+u=t\,e^{\frac{i3\pi}{2}}$ L_2: L H S of the branch cut $(0<t<\infty)$

$-k+u=\epsilon\, e^{i\theta}$ C_ϵ: Around the branch point $(-\frac{\pi}{2}<\theta<\frac{3\pi}{2})$

$-k+u=R\ e^{i\theta}$ C_R: On the semi-circle $(-\pi<\theta<0)$

In problem (1.5), the pole is α. However, because the pole α does not fit in our curve, $Rez=0$. The integral $I(\alpha)$ has a branch point at $u=k$. The integrals will be expressed as

$$\frac{1}{2\pi i}\int_{L_+} f(u)du+\frac{1}{2\pi i}\int_{C_R} f(u)du+\frac{1}{2\pi i}\int_{L_1} f(u)du+\frac{1}{2\pi i}\int_{C_\epsilon} f(u)du$$
$$+\frac{1}{2\pi i}\int_{L_2} f(u)du=0\quad, \tag{3.3}$$

where the function f is defined as

$$f(u)=\frac{H_0^{(2)}(Kb)e^{-iuc}}{KH_1^{(2)}(Ka)(u-\alpha)}\quad.$$

Then, we obtain

$$\frac{1}{2\pi i}\int_{C_R} f(u)du = \frac{1}{2\pi i}\int_{C_R} f\left(k + Re^{i\theta}\right) iRe^{i\theta}d\theta, \quad as \quad R \to \infty \quad ,$$

Where

$$-k + u = R \;\; e^{i\theta} \quad \to u = \; k + \;\; R \;\; e^{i\theta}$$

and

$$K = \sqrt{k^2 - u^2} \qquad ,$$

$$K = \sqrt{(k-u)(k+u)} \;,$$

$where\ K \to \infty$.

By putting the values of *u* in the equation *K*, we get

$$K = \sqrt{-u^2\left(1 - \frac{k^2}{u^2}\right)} \quad then \quad K \quad \approx -i|u| \;\; \approx -iR \tag{3.4}$$

In the asymptotic expansion of Hankel function relating to large arguments, it is possible to use the asymptotic.

$$H_0^{(2)}(Kb) \approx \sqrt{\frac{2}{\pi Kb}}e^{-iKb+\frac{i\pi}{4}} \tag{3.5}$$

$$H_1^{(2)}(Ka) \approx \sqrt{\frac{2}{\pi Ka}}e^{-iKa+\frac{i3\pi}{4}} \tag{3.6}$$

[17] .

From equations (3.4), (3.5), and (3.6), we obtain

$$\frac{1}{2\pi i}\int_{C_R} f(u)du = \frac{1}{2\pi i}\int_{CR} \frac{\sqrt{\frac{2}{\pi K b}}\, e^{-iKb+\frac{i\pi}{4}} \quad e^{-i\left(k+R\, e^{i\theta}\right)c}}{K\, \sqrt{\frac{2}{\pi K a}}\, e^{-iKa+\frac{i3\pi}{4}} \quad \left(k + Re^{i\theta} - \alpha\right)}\; iR\, e^{i\theta} d\theta$$

$$\frac{1}{2\pi i}\int_{C_R} f(u)du = \frac{-i}{2\pi}\sqrt{\frac{a}{b}}\int_{C_R} \frac{e^{-iK(b-a)-i\left(k+R\, e^{i\theta}\right)c}}{K \quad \left(k + Re^{i\theta} - \alpha\right)}\; R\, e^{i\theta} d\theta \qquad (3.7)$$

In equation (3.7), we must find the part ($e^{-iK(b-a)-i\left(k+R\, e^{i\theta}\right)c}$)
and the range $-\pi < \theta < 0$.

$e^{-iK(b-a)-i\left(k+R\, e^{i\theta}\right)c}$, c is positive for the interval $(0 < \theta_0 < \frac{\pi}{2})$,

which implies that

$$\lim_{R\to\infty} e^{-R(b-a)-ikc-iRc\, cos\theta+Rc\, sin\theta} = 0\, ,$$

Then, we obtain

$$\frac{1}{2\pi i}\int_{C_R} f(u)\, du = 0\, .$$

Then where (c_ϵ) around the branch point, we have,

$$\frac{1}{2\pi i}\int_{C_\epsilon} f(u)\, du = \frac{1}{2\pi i}\int_{C_\epsilon} f\left(k + \epsilon e^{i\theta}\right)\; i\epsilon e^{i\theta} d\theta \quad (\epsilon \to 0)$$

From figure (3.3), we assume

$$k - u = -\epsilon\, e^{i\theta}$$

$$k + u = 2k + \epsilon\, e^{i\theta}$$

$$u = k + \epsilon\, e^{i\theta} \to du = i\, \epsilon\, e^{i\theta} d\theta$$

$$K = \sqrt{(2k + \epsilon\, e^{i\theta})(-\epsilon\, e^{i\theta})} \quad \to K \approx -i\sqrt{2k\, \epsilon\, e^{i\theta}}$$

By asymptotic behavior when $Kb \to 0$

$$H_0^{(1,2)}(Kb) \approx \pm 2i \frac{log\ \ (Kb)}{\pi} \tag{3.8}$$

$$H_n^{(1,2)}(Kb) \approx \pm i(n-1)!\ \frac{\left(\frac{Kb}{2}\right)^{-n}}{\pi} \quad ,\ \ n = 1,2, \dots \dots .. \tag{3.9}$$

From (3.8) and (3.9), we obtain

$$H_0^2(Kb) \approx \frac{-2i}{\pi} log\ \ (Kb) \quad ,$$
$$H_1^2(Ka) \approx \frac{2i}{\pi}\ \frac{1}{Ka} \qquad\qquad ,$$

and

$$\frac{1}{2\pi i}\int_{C_\epsilon} f(u)du = \frac{1}{2\pi i}\int_{C_\epsilon} \frac{\frac{-2i}{\pi} log\ \ (Kb) \qquad e^{-i\left(k+\ \epsilon\ e^{i\theta}\right)c}}{K\ \frac{2i}{\pi}\ \frac{1}{Ka} \quad (k+\ \ \epsilon\ e^{i\theta} - \alpha)} i\sqrt{2k\epsilon e^{i\theta}}\ d\theta$$

$$= \frac{-a}{2\pi}\int_{C_\epsilon} \frac{log\ \ (Kb)\ \ \ e^{-i\left(k+\ \epsilon\ e^{i\theta}\right)c}}{(k+\epsilon e^{i\theta} - \alpha)}\ i\sqrt{2k\epsilon e^{i\theta}} d\theta \quad . \tag{3.10}$$

In (3.10), we must find $\lim\limits_{x\to 0} \sqrt{\epsilon}\ \ log(Kb)$

By using the fact

$$\lim_{x\to 0} x\ log(x) \ \ = 0\ ,$$

we obtain

$$\lim_{x\to 0} \sqrt{\epsilon}\ \ \ log(Kb) \quad = 0\ \ .$$

By using the above properties, the result of integration becomes

$$\frac{1}{2\pi i}\int_{C_\epsilon} f(u)du \;=0\;\;.$$

After rearranging it in (3.3), we get

$$\frac{1}{2\pi i}\int_{L_+} f(u)du+\frac{1}{2\pi i}\int_{L_1} f(u)du+\frac{1}{2\pi i}\int_{L_2} f(u)du=0\;\;,$$

$$I(\alpha)=-\frac{1}{2\pi i}\int_{L_1} f(u)du-\frac{1}{2\pi i}\int_{L_2} f(u)du\quad. \tag{3.11}$$

From analytical continuation property of Hankel functions given in [17], we have:

$$H_0^{(2)}\left(e^{-i\pi}Kb\right)=-H_0^{(1)}(Kb)\;, \tag{3.12}$$

$$H_1^{(2)}\left(e^{-i\pi}Ka\right)=H_1^{(1)}(Ka)\quad. \tag{3.13}$$

On the RHS of branch cut, we obtain:

$$k-u=-t\,e^{\frac{-i\pi}{2}}\;\;,$$
$$k+u=2k-it\,.$$

and

$$K^*=\sqrt{k^2-u^2}\;\;=\sqrt{(k-u)(k+u)}\;\;=\;\sqrt{t\,e^{\frac{-i\pi}{2}}(it-2k)}\,. \tag{3.14}$$

On the LHS of branch cut, we obtain:

$$k-u=-t\,e^{\frac{i3\pi}{2}}\;\;,$$
$$k+u=2k-it\;\;.$$

Henceforth,

$$K = \sqrt{k^2 - u^2} \quad = \sqrt{(k-u)(k+u)} = \sqrt{t\, e^{\frac{i3\pi}{2}}(it-2k)}\,. \tag{3.15}$$

Then, from equations (3.14), and (3.15), we get

$$K^* = e^{-i\pi}K\,. \tag{3.16}$$

As a result of (3.16), in the following integral

$$I(\alpha) = -\frac{1}{2\pi i}\int_{L1} \frac{H_0^{(2)}(K^*b)e^{-iuc}}{K^*H_1^{(2)}(K^*a)(u-\alpha)}du - \frac{1}{2\pi i}\int_{L2} \frac{H_0^{(2)}(Kb)e^{-iuc}}{KH_1^{(2)}(Ka)(u-\alpha)}du\,,$$

on integral line L_1, we get $e^{-i\pi}K$ while on integral line L_2 we get K. It yields that

$$I(\alpha) = -\frac{1}{2\pi i}\int_{L1} \frac{H_0^{(2)}\left(e^{-i\pi}Kb\right)e^{-iuc}}{e^{-i\pi}KH_1^{(2)}(e^{-i\pi}Ka)(u-\alpha)}du - \frac{1}{2\pi i}\int_{L2} \frac{H_0^{(2)}(Kb)e^{-iuc}}{KH_1^{(2)}(Ka)(u-\alpha)}du.$$

By using (3.12) and (3.13), we obtain

$$\begin{aligned} I^{bc}(\alpha) = &-\frac{1}{2\pi i}\int_{\infty}^{0} \frac{H_0^{(1)}(Kb)e^{-i(k-it)c}}{KH_1^{(1)}(Ka)(k-it-\alpha)}(-i)dt \\ &-\frac{1}{2\pi i}\int_{0}^{\infty} \frac{H_0^{(2)}(Kb)e^{-i(k-it)c}}{KH_1^{(2)}(Ka)(k-it-\alpha)}(-i)dt \end{aligned}$$

It implies that

$$I^{bc}(\alpha) = \frac{1}{2\pi}\int_{0}^{\infty} \frac{e^{-i(k-it)c}}{(k-it-\alpha)}\left[\frac{H_0^{(2)}(Kb)}{KH_1^{(2)}(Ka)} - \frac{H_0^{(1)}(Kb)}{KH_1^{(1)}(Ka)}\right]dt \quad .$$

Defining a function p (t) as

$$p(t) = \left[\frac{H_0^{(2)}(Kb)}{KH_1^{(2)}(Ka)} - \frac{H_0^{(1)}(Kb)}{KH_1^{(1)}(Ka)}\right] e^{-i(k-it)c} \quad ,$$

we obtain a semi-infinite convergent integral

$$I^{bc}(\alpha) = \frac{1}{2\pi} \int_0^{\infty} \frac{p(t)}{(k - it - \alpha)} \, d\,t \quad , \tag{3.17}$$

which is suitable for numerical calculations.

CHAPTER 4

SMOOTH CONTOUR INTEGRATION

The function $Q(u)$ is a complex-valued function in the complex plane, and also along the real axis in particular. We are concerned with the evaluation of $I(\alpha)$ in (1.5) by using $Q(u)$. Along the real axis, the imaginary part of the wave number k is zero [6], [18]. $Q(u)$ is given by

$$Q(u) = \frac{H_0^{(2)}(Kb)}{KH_1^{(2)}(Ka)}\, e^{-iuc} \tag{4.1}$$

and $I(\alpha)$ is defined as in (1.5) in terms of $Q(u)$ over the contour L_+ which is previously presented in figure (1.1). Then

$$I(\alpha) = \frac{1}{2\pi i}\int_{L+} \frac{Q(u)}{u-\alpha} du \tag{4.2}$$

When the imaginary part of k is zero, which is the case in reality, integral of $I(\alpha)$ turns to an integral from $-\infty\ to + \infty$. Now we can rearrange the integral (4.2) as

$$\begin{aligned}
\lim_{p\to\infty} \frac{1}{2\pi i}\int_{-P}^{P} \frac{Q(u)}{u-\alpha} du &= \lim_{p\to\infty} \frac{1}{2\pi i}\int_{-P}^{0} \frac{Q(u)}{u-\alpha} du + \lim_{p\to\infty} \frac{1}{2\pi i}\int_{0}^{P} \frac{Q(u)}{u-\alpha} du \\
&= \lim_{p\to\infty} \frac{1}{2\pi i}\int_{0}^{P} \frac{Q(-u)}{-u-\alpha} du + \lim_{p\to\infty} \frac{1}{2\pi i}\int_{0}^{P} \frac{Q(u)}{u-\alpha} du \\
&= \lim_{p\to\infty} \frac{1}{2\pi i}\int_{0}^{P} \left\{\frac{Q(u)}{u-\alpha} - \frac{Q(-u)}{u+\alpha}\right\} du \\
&= \frac{1}{2\pi i}\int_{0}^{\infty} \left\{\frac{Q(u)}{u-\alpha} - \frac{Q(-u)}{u+\alpha}\right\} du
\end{aligned}$$

to ensure the final form of the integral is convergent. Therefore, $I(\alpha)$ can be written as follows:

$$I(\alpha) = \frac{1}{2\pi i}\int_0^\infty \left\{\frac{Q(u)}{u-\alpha} - \frac{Q(-u)}{u+\alpha}\right\} du \quad .$$

Explicitly, it can be rewritten as

$$I(\alpha) = \frac{1}{2\pi i}\int_0^\infty \frac{H_0^{(2)}(Kb)}{KH_1^{(2)}(Ka)} \left[\frac{e^{-iuc}}{u-\alpha} - \frac{e^{iuc}}{u+\alpha}\right] du \quad , \tag{4.3}$$

where it can be shown that it is a convergent integral. But this form is still not suitable for a numerical calculation due to the existence of a possible pole on the real axis. Therefore, to evaluate the numerical solution of $I(\alpha)$, we introduce the following variable change to get rid of possible zeros or poles on the real axis:

$$u = \xi\,(t),\;\; \xi\,(t) = t + id\frac{4t/q}{3+(t/q)^4} \qquad 0 \le t < \infty \quad . \tag{4.4}$$

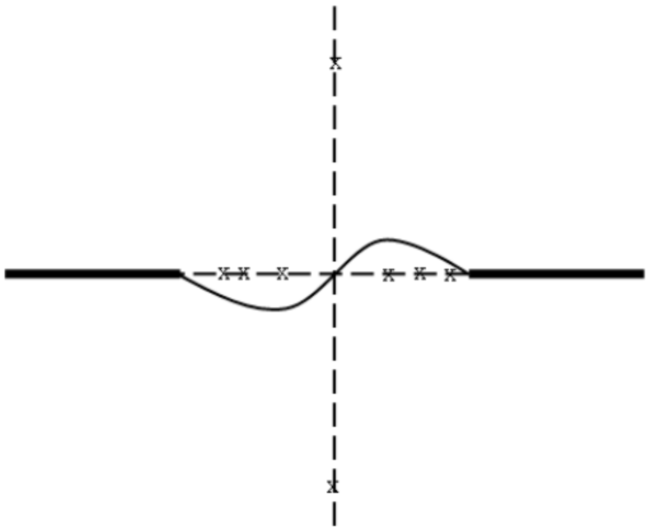

Figure 4.1. Sketch of deformed integration contour in complex plane [19].

The parameters $d > 0$ and $q > 0$ determine the height and width of the contour, respectively. However, it is no matter what is picked randomly. On the other hand, when deforming original contour L_+ to smooth contour in (4.4) one must be careful about if any pole is crossed.

Although integral in (4.3) is calculated on a contour that does not go through poles and zeros, it is still not in a desirable form in terms of numerical computation since the integration interval is infinite. For this reason, we use another variable change to make the boundary of the integral finite [6], [18]. Changing the following variable converts the integration interval to finite interval of [0,1], we obtain

$$t = \eta\,(s)\,, \qquad \eta\,(s) = \frac{s}{(1-s)^2} \quad , \; 0 \le s \le 1 \; . \tag{4.5}$$

By using the transformation of (4.4) and (4.5), we can now define $I(\alpha)$ as

$$I(\alpha) = \frac{1}{2\pi i}\int_{L+} \frac{Q(u)}{u-\alpha}\,du = \frac{1}{2\pi i}\int_0^1 h(s,\alpha)\xi'\big(\eta(s)\big)\,\eta'(s)ds \qquad , \tag{4.6}$$

Where

$$h(s,\alpha) = \frac{Q(\xi\,(\eta(s)))}{\xi\,\big(\eta(s)\big)-\alpha} - \frac{Q(-\xi\,(\eta(s)))}{\xi\,\big(\eta(s)\big)+\alpha} \;\; ,$$

$$\eta'(s) = \frac{s}{(1-s)^2} \;\; ,$$

$$\xi'\big(\eta(s)\big) = \frac{s}{(1-s)^2} + id\,\frac{\frac{4s}{(1-s)^2}/q}{3+(\frac{s}{(1-s)^2}/q)^4} \quad .$$

The integral formula in (4.6) is now very convenient for numerical calculations. If any pole is crossed residue contribution must be calculated.

CHAPTER 5

NUMERICAL CALCULATIONS

In this chapter, integral solutions obtained by the formulas (2.9), (3.17) and (4.6) for $\boldsymbol{I(\alpha)}$ in different chapters are going to be evaluated numerically on computer to compare different solution approaches. This will also allow us to understand the behaviour of $\boldsymbol{I(\alpha)}$ for different integral parameters. Since $\boldsymbol{I(\alpha)}$ is a complex valued function absolute value $|\boldsymbol{I(\alpha)}|$ versus frequency for different values of parameters $\boldsymbol{a, b, c}$ will be plotted. Frequency is related with the wave number by the formula $\boldsymbol{k = 2\pi f/c}$ and *c* is the speed of sound taken as 340m/s. The value of $\boldsymbol{\alpha}$ was chosen as $\boldsymbol{k/2}$ on the real interval $(\boldsymbol{-k, k})$.

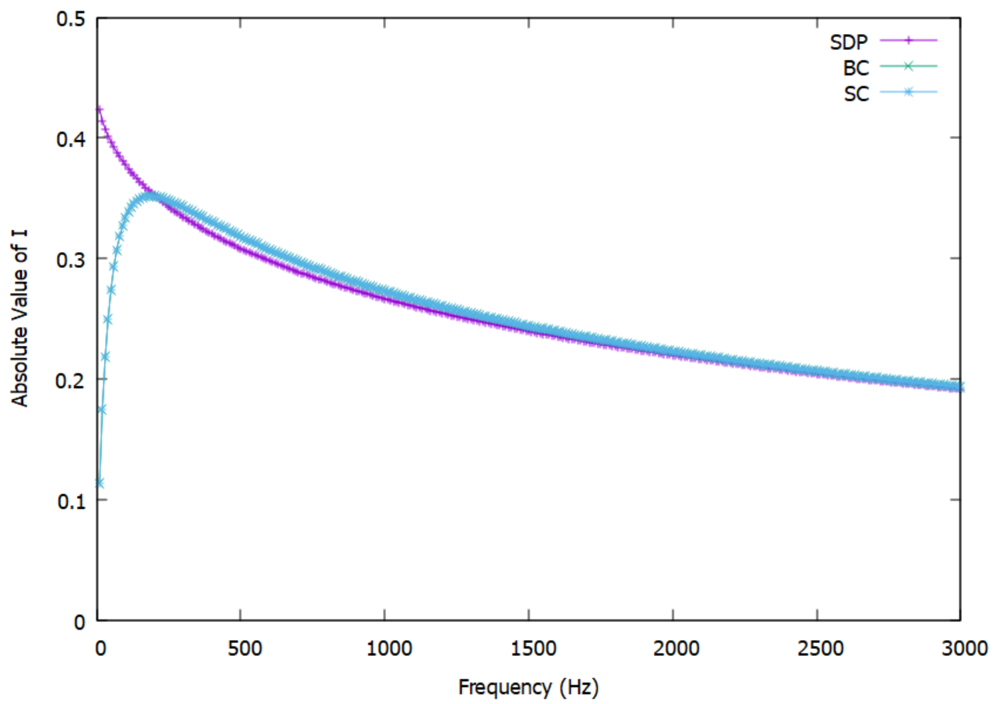

Figure 5.1. $|\boldsymbol{I(\alpha)}|$ **versus frequency for** $\boldsymbol{a = 0.5}$, $\boldsymbol{b = 1}$, $\boldsymbol{c = 0.5}$.

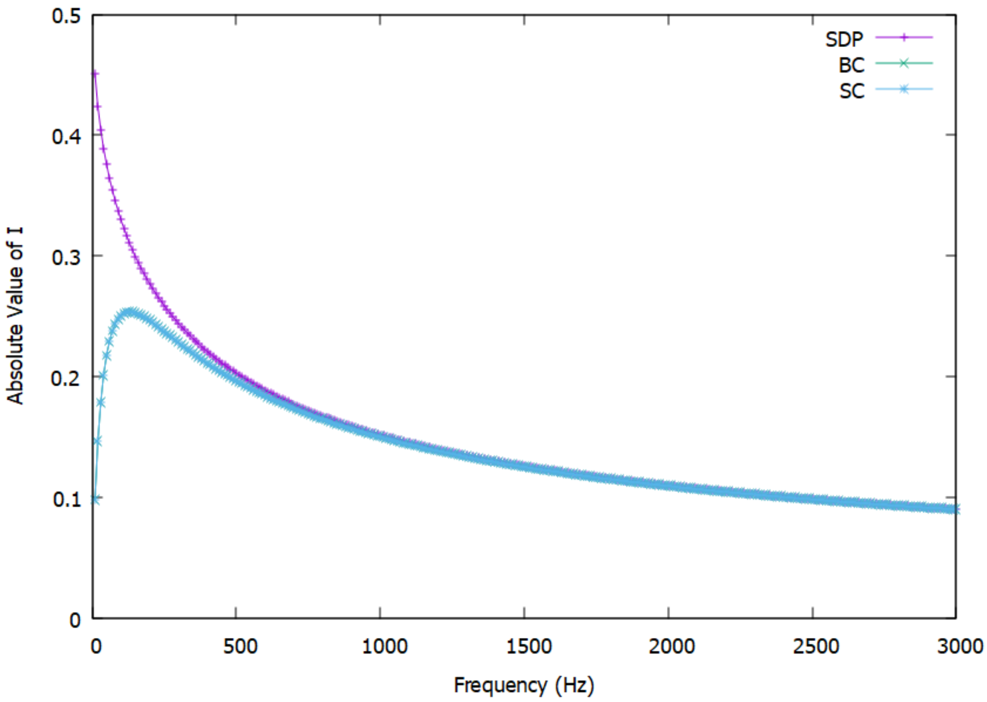

Figure 5.2. $|I(\alpha)|$ **versus frequency for** $a = 0.5$, $b = 1$, $c = 1$.

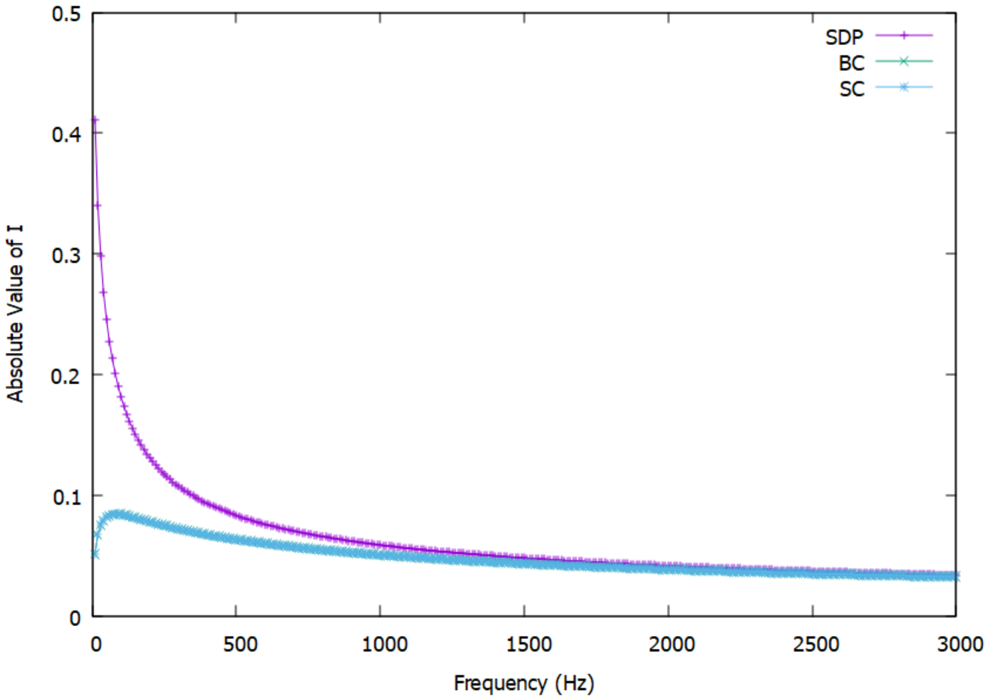

Figure 5.3. $|I(\alpha)|$ **versus frequency for** $a = 0.5$, $b = 1$, $c = 5$.

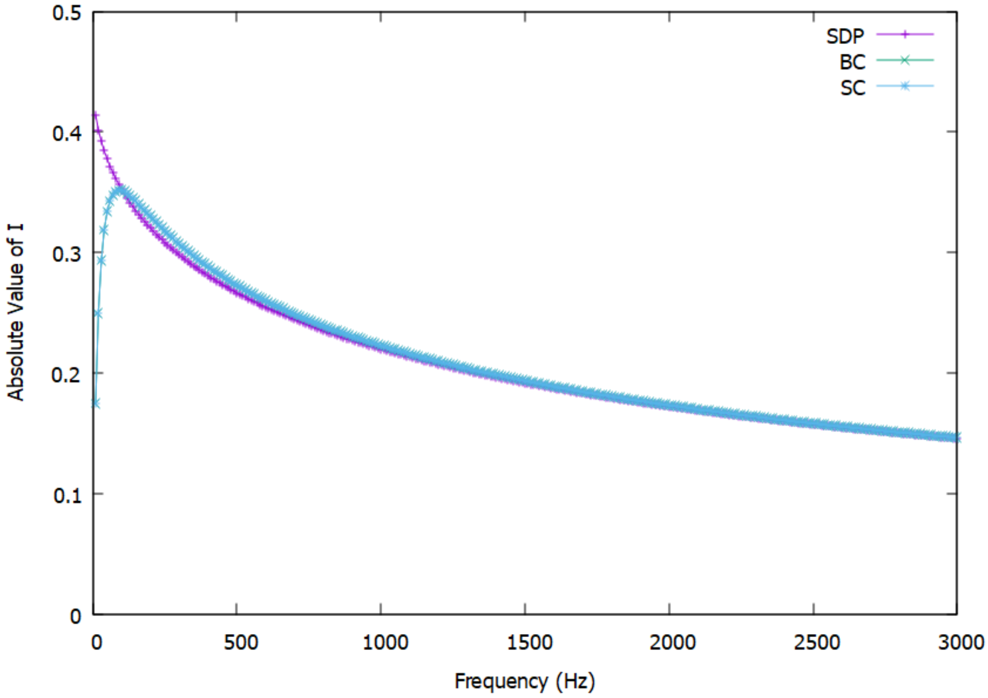

Figure 5.4. $|\boldsymbol{I}(\boldsymbol{\alpha})|$ **versus frequency for** $\boldsymbol{a} = \boldsymbol{1}$ **,** $\boldsymbol{b} = \boldsymbol{2}$ **,** $\boldsymbol{c} = \boldsymbol{1}$.

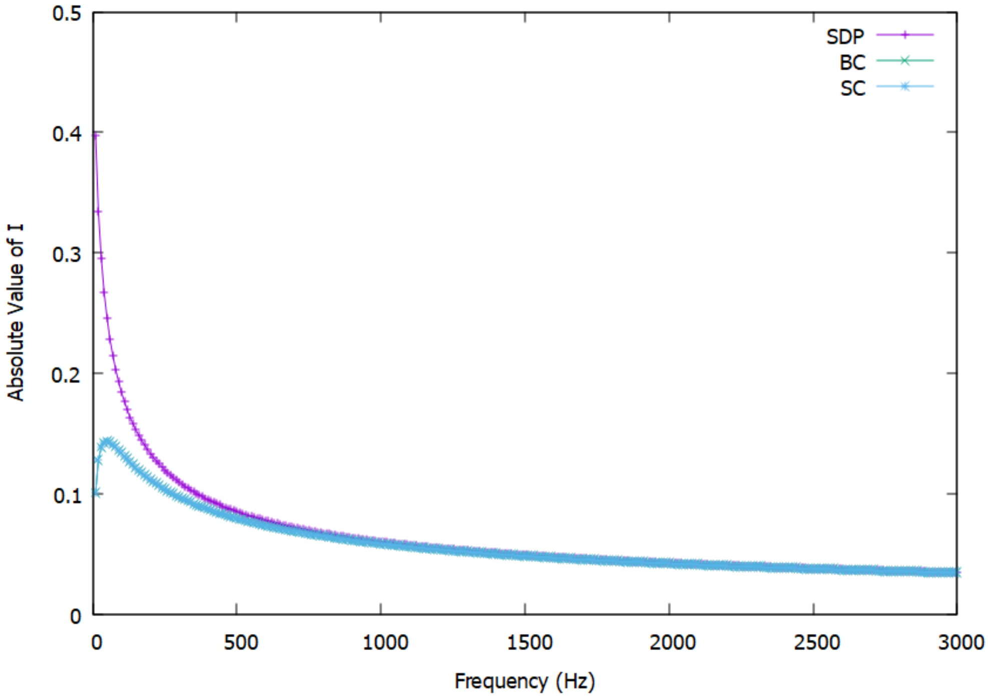

Figure 5.5. $|\boldsymbol{I}(\boldsymbol{\alpha})|$ **versus frequency for** $\boldsymbol{a} = \boldsymbol{1}$ **,** $\boldsymbol{b} = \boldsymbol{2}$ **,** $\boldsymbol{c} = \boldsymbol{5}$.

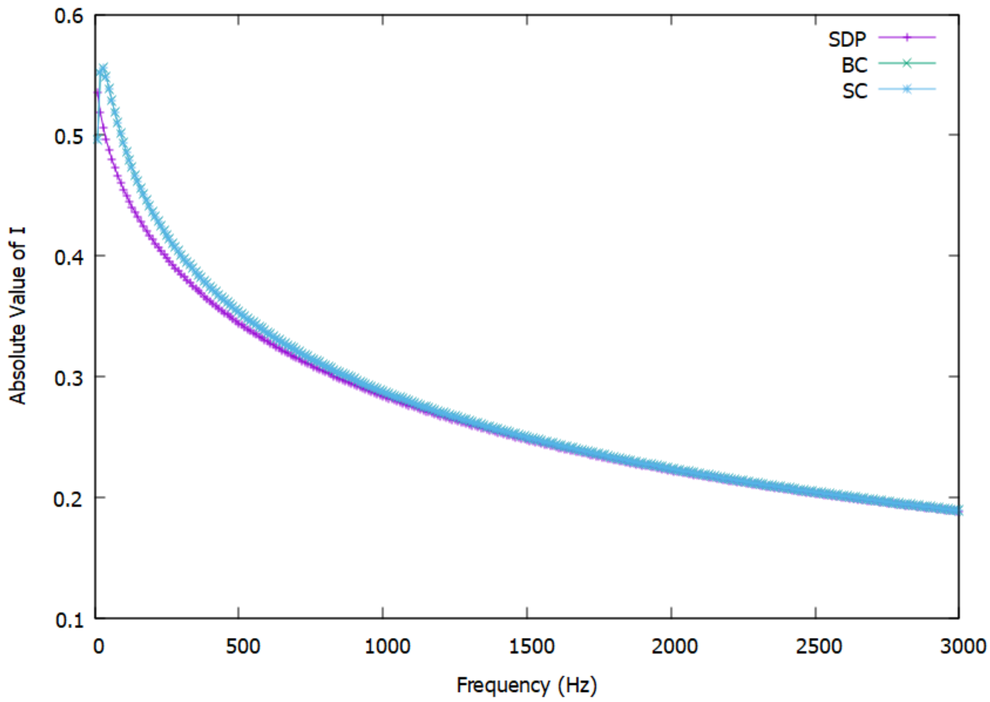

Figure 5.6. $|\boldsymbol{I}(\boldsymbol{\alpha})|$ **versus frequency for** $\boldsymbol{a} = \mathbf{5}\,,\ \boldsymbol{b} = \mathbf{6}\,,\ \boldsymbol{c} = \mathbf{1}$.

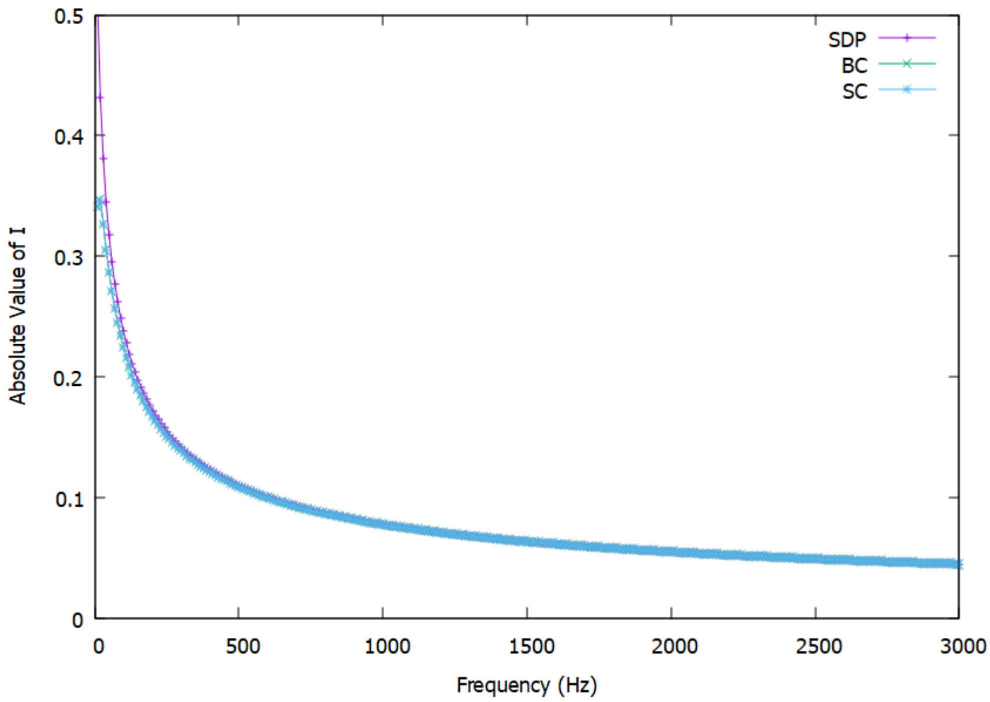

Figure 5.7. $|\boldsymbol{I}(\boldsymbol{\alpha})|$ **versus frequency for** $\boldsymbol{a} = \mathbf{5}\,,\ \boldsymbol{b} = \mathbf{6}\,,\ \boldsymbol{c} = \mathbf{5}$.

In all figures one can see that Branch cut (BC) and Smooth contour (SC) solutions are nearly the same for any frequency. On the other hand, the saddle point technique (SDP) overlaps with the other two solutions at increasing frequencies as expected. The reason why the SDP behaves differently from the other two methods at low frequencies is the asymptotic formulas used in the process of technique, which are only valid at large argument values (namely at high frequencies). But, when we increase all parameter values as in Fig 5.7 we observe that even for the low frequencies all methods produce similar values.

It is also observed in all figures that $|\boldsymbol{I}(\boldsymbol{\alpha})|$ tends to a constant value for increasing values of frequency. In case the frequencies near zero we expect that the integral tends to zero as can be seen in the numerical solutions of BC and SC methods.

Finally, when we compare all three methods, although the Bc and SC methods produce close results at all frequencies, the relative simplicity of the SC method in analytical formulation and its convenience in numerical calculation (in terms of being a finite integral) make it more advantageous than the other two methods. Moreover, SDP can be preferred at higher frequencies as it does not require numerical integral calculation and for its analytical solution.

CHAPTER 6

SUMMARY

In the present study, we worked on solving certain types of complex integrals by various methods. Asymptotic and analytical approaches which involve contour deformation on complex plane have been used to examine these complex integrals. For examining purpose and to give an example for readers, a certain kind of complex integral was solved in three different approaches. Each of our three solutions exhibited a separate solution of a different nature. One of the approaches we have used to solve the proposed problem is the saddle point technique, which is an asymptotic evaluation. The resulting solution is analytic but valid for larger values of Ω (in our example it is wave number k). It means that the greater value of Ω is, the closer our solution to the correct value. Our consequent way of solution was an evaluation by transforming L onto the branch cut. In this approach, integration contour is deformed onto branch cuts by using the Cauchy theorem. Using the analytical continuity property of Hankel functions on different sides of the branch cut, where square root function $K(\alpha)$ has opposite values, the integral turned into a semi-infinite convergent integral. In the last method of solution, integration contour is deformed onto a smooth contour defined by some parameter. After that integration interval reduced from infinite interval $(-\infty, \infty)$ to finite interval $[0,1]$ by using a change of variable. The resulting integral is finite and has a more convenient form for numerical calculations.

To better understand each solution approach and for a comparison, numerical calculations for different integral parameters such as a, b, c, k have been done. As a result of these studies, it was seen that all methods produced compatible results especially for high frequencies. For low frequencies, BC and SC methods seen to be Compatible while SDP mismatches as espceted.

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RESUME

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