INVERSE SEMIGROUPS AND ITS RELATION WITH LEAVITT PATH ALGEBRA

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I certify that in my opinion the thesis submitted by Mounzer Khıf MLEHAN ALJADAAN titled “INVERSE SEMIGROUPS AND ITS RELATION WITH LEAVITT PATH ALGEBRA” is fully adequate in scope and in quality as a thesis for the degree of Master of Science.

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Mounzer Khlıf MLEHAN ALJADAAN

ABSTRACT

M. Sc. Thesis

INVERSE SEMIGROUPS AND ITS RELATION WITH LEAVITT PATH ALGEBRA

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Institute of Graduate Programs
The Department of Mathematics

Thesis Advisor:
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In this thesis, we consider an inverse semigroup class constructed from Leavitt path algebras. In the beginning of the thesis, theoretical background on directed graphs and its properties are discussed. Then it continues with important theories and definitions of the Leavitt path algebras. Furthermore, all the theories are supported with good examples. In the following, we examined the role of inverse semigroups in algebra and investigated its structures, ideals and homomorphisms in details. In this thesis, we especially give our attention to analyze the class of inverse semigroups related to the Leavitt path algebras. We studied a presentation for the Leavitt inverse semigroups and defined the structure of the Leavitt inverse semigroups.

Key Words : Semi Groups, Inverse Semi groups, Graph Theory, Leavitt Path Algebra, ideals.

Science Code:

ÖZET

Yüksek Lisans Tezi

TERS YARI GRUPLAR VE LEAVITT YOL CEBIRLERI ILE ILIŞKILERI

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Bu tezde, Leavitt yol cebirlerinden oluşturulmuş bir ters yarıgrup sınıfını ele alıyoruz. Tezin başlangıcında yönlendirilmiş çizgeler ve özellikleri ile ilgili teorik bilgiler çalıştik, devamında Leavitt yol cebirleri hakkında önemli teorilere ve tanımlara yer verildik. Ayrıca tüm teorileri iyi örneklerle destekledik. Tezin devamında, ters yarı grupların cebirdeki rolünü inceledik ve bu cebirsel yapının ideallerini ve homomorfizmlerini ayrıntılı olarak inceledik. Bu tezi özellikle çalımamızda ki amac, Leavitt yol cebirleri ile ilgili ters yarıgruplar arası ıilişkiyi incelemektir. Bu sebeple, Leavitt ters yarıgruplarının tanımı üzerinde çalıştık ve Leavitt ters yarıgruplarının yapısını inceledik.

Anahtar Kelimeler: Yarı Gruplar, Ters Yarı gruplar, Çizge Teorisi, Leavitt Yol Cebirleri, idealler.

Bilim Kodu :

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SYMBOLS

E	: A directed graph.
r, s: $E^1 \rightarrow E^0$	: Functions.
e	: Edge.
u, v	: Vertices.
E^0	: A set of vertices.
E^1	: A set of edges.
s (e)	: Source of edge.
r (e)	: Range of edge.
P	: A path.
s (p)	: Source of p.
$r(p)$	: Range of p.
$L_K(E)$	: Leavitt path algebras,
K	: Field.
R	: Ring.
I (X)	: The ideal of R.
K	: Field.
IBN	: Invariant Base Number.
(m,n)	: Module.
$\ell(\mu)$, $\mid \mid$	: The length of path μ.
E1	: Source set of edge.
E2	: Range set of edge.
CK1	: Source set of vertexs.
CK2	: Sange set of vertexs.
$CSP(v)$	: Simple closed paths based on v.
$K [x, x^{-1}]$	: Laurent polynomial is a K-algebra.
E_T	: Toeplitz graph.

$\bar{X}$	: The hereditary-saturated closure of X.
S	: Inverse semigroup.
s^{-1}	: A inverse of S.
T(v)	*:* Tree of *v*.
Dom$f$	: Domain of f.
Imf	: Image of f.
0_{BA}	: A unique empty partial function.
1_X	: The identty function on $X \subseteq$A.
1_A	: The identty function 1_A on A.
$Fg = fog$	: Composite a partial function.
S^1	: Inverse monoid.
S^o	: Inverse zero.
sa $\in$A	: Ieft ideals.
as $\in$ A	: Right ideals.
sS	: the Basic right ideal.
Ss	: Basic right ideal.
SsS	: The basic two-sided ideal.
(K, $\leq$)	: A partially ordered set.
E(S)	: An order ideal of S.
(E(S), $\leq$)	: A meet semilattice.
(NE)	: No exits.
v, p, p^*	: Elements of Leavitt path algebra.
L_F (E)	: Leavitt path algebra.
B_X(G)	: Brandt semigroup.
Δ	: Connected graphs.

PART 1

INTRODUCTION

Leavitt path algebra is an algebraic structure and constructed from a directed graph. This algebra generalizes the Laavitt algebras and it has similary construction with the graph C^*-algebras. For a directed graph E and field K, $L_K(E)$ Leavitt path algebra is first defined in 2005 by G. Abrams and G. Aranda Pino as a generalization of Leavitt algebras and expanded to arbitrary graphs in 2008 [14-15]. In the following years, many researchers interested the infinitely simple properties, socle, finite dimensional structures, prime and maximal ideals of $L_K(E)$ [16], [17], [18]. In 2011, M. Tomforde examined the $L_R(E)$ Leavitt path algebras for a finite directed graph E and commutative ring R. He defined necessary an d sufficient conditions for $L_R(E)$ to be basically simple [19]. In 2015, H. Larki $L_R(E)$ extended Leavitt path algebras to countable graphs and studied the characterization of prime and primitive ideal structures with $L_R(E)$ being prime and primitive rings. [20].

Algebraic structural information about Leavitt path algebras can be obtained from the theory of inverse semigroups. Recently, inverse semigroups became increasingly important in algebraic natures. In [8] authors studied on inverse semigroups which is obtained from the Leavitt inverse semigroups. They observed that inverse semigroups are related to the Leavitt path algebras. By the this study, they showed that the Leavitt inverse semigroups can be represent by a graph in terms of generators and relations. Considering this study, we firtsly give our attention to understand algebraic structure of the Leavitt path algebra and inverse semigroups and then observed a relation between them. Therefore, this thesis is organized as follows:
In part 2, we considered history of Leavitt path algebra and inverse semigroups with some way and technical of the solving problems, with using some references to be sure about the history of our subjects.

In part 3, graph theory has an importance part in the applied and pure algebra, So, we explained graph theory with its important definitions, examples, and theories. One of the importance of this section is nature of the Leavitt path algebra. In this part we firstly studied basic properties of the Leavitt path algebras and supported them with good examples, then we examined the ideal of the Leavitt path algebras. Throughout the study of the Leavitt path algebras, the main book we followed was Leavitt path Algebras written by G. Abrams, P. Ara and M. Molina [1].

In part 4, our first aim is to study inverse semigroups in detail to better understand the next section. Thus, we examined all the nature of the inverse semigroups. In the beginning of this section, we reminded the partial bijections and then give the definition of the inverse semigroups. In the following, we stated the importance of the ordered groupoids. Then inverse properties, ideals, natural partial orders, compatibility relations, meets and joins and also homomorphisms between inverse semigroups are studied in detail. Theories and definitions of all subjects are emphasized, and their importance is mentioned. Throughout the section, we follow [2].

In part 5, the main purpose of this thesis study was to examine the relationship between inverse semigroups and Leavitt path algebras. Therefore, in this section we gave our consideration to the paper [8]. Considering this study, we have learned about the relationship between these two algebraic structures, and we have outlined the important theories and results for this study in this part.

In part 6, we concluded the thesis by stating the purpose and importance of the studies.

PART 2

LITERATURE REVIEW

V. V. Wagner defined inverse semigroups for the first time in 1952 [3], by the meanwhile this study also introduced by Gordon Preston in 1954 [4], [9] and [10]. In the following years many researchers interested in inverse semigroups [5], [7] and [6], [11] [12]. In the 1940s and 1950s Charles Ehresmann considered this theory with a different mathematical perspective. Inverse semigroups were introduced as part of the legacy of the Klein's Erlanger Program and Lie's theory of infinite continuous groups.

The Leavitt path algebra was initially introduced in 2007 by Ara, Moreno and Pardo [13], and almost simultaneously but independently by Abrams and Pino [14]. Nearly a decade later, this algebra has attracted considerable attention not only ring theorists, but also C∗ -algebras and group theorists. In 2005, G. Abrams and G. A. Pino defined the Leavitt path algebra, $L_K(E)$, of a finite graph E with coefficients from a field K as a generalization of the Leavitt path algebras and expanded it to the arbitrary graph in 2008. In 2015 G. Aranda Pino, K.M. Rangaswamy and M. Siles Molina obtained that the Leavitt path algebra is a right R-module on itself. Also, they observed that Endomorphism ring of the Leavitt path algebras are von Neumann regular.

Structural knowledge of Leavitt path algebras can be obtained from the theory of inverse semigroups. Therefore, J. Meakin, David Milan and Zhengpan Wang studied a class of inverse semigroup which is obtained by the Leavitt path algebras in 2021 [8]. They observed that these semigroups are also related to the graph inverse semigroups.

PART 3

PRELIMINARIES

In the first section, we give the necessary background of the graph theory. For the convenience of the reader, we also give the necessary examples of the graph. We refer reader to [15], [16], [18] for more details about the graph theory. In the following of this chapter, we also discuss the Leavitt path algebra. The main references for the second section are [1], [13] and [14].

3.1. PRELIMINARIES ON GRAPH THEORY

Interest in graph theory and its applications has grown rapidly in the last two decades. The reason for this increase is that we can find solutions to many problems in our daily life with graph theory. Many situations we encounter can be described by a set of points and diagrams of lines connecting these points. Graph is also used in algebraic structures, constructing graph group and ring graphs, and explaining various algebraic properties. This chapter contains background information on the graph theory.

3.1.1. Definition

A graph E is an ordered pair $E = (E^0, E^1)$ comprising:

- E^0, a set of vertices
- E^1, a set of edges which are unordered pairs of vertices

3.1.2. Definition

A directed graph $E = (E^0, E^1, r, s)$ consists of two sets E^0, E^1, and two functions r, s: $E^1 \rightarrow E^0$. For any $e \in E^1$, $s(e)$ is called the source of *edge and* $r(e)$ is called the range of edge. Let $e \in E^1$ and $v, w \in E^0$, *then* $s(e)$ is called source of e and $r(e)$ is called range of e. For any two edges $e_1, e_2 \in E^1$, if $r(e_1) = s(e_2)$, then e_1, e_2 are called adjacent edges.

3.1.3. Definition

For a vertex u, $s^{-1}(u)$ is the set of edges with source u *and* $r^{-1}(u)$ is the set of edges with a range. The vertex u which $s^{-1}(u) = \emptyset$ is called the sink, and $r^{-1}(u) = \emptyset$ is called the source. A vertex which is both a source and a sink is called isolated.

3.1.4. Example

[1] In the following graph E, the vertices of u_1, u_2, v_2, and v_3 are sink, and the vertex u is a source

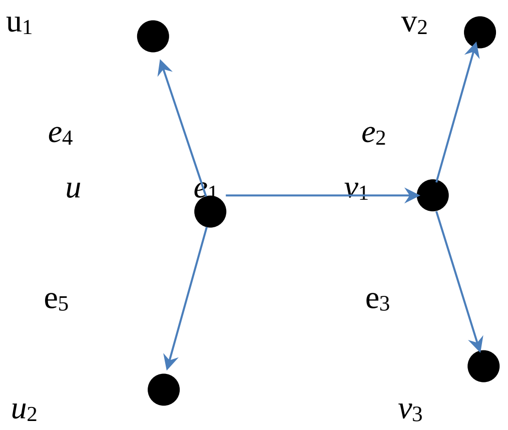


Figure 3.1. Directed graph.

Moreover, $s(e_1) = u$, $r(e_1) = v_1$, $r(e_1) = v_1 = s(e_2) = s(e_3)$, $s^{-1}(u) = \{e_1, e_4, e_5\}$, $r^{-1}(u) = \emptyset$, $s^{-1}(v_1) = \{e_2, e_3\}$, $r^{-1}(v_1) = \{e_1\}$, $r^{-1}(v_3) = \{e_3\}$, $s^{-1}(v_3) = \emptyset = s^{-1}(v_2)$, $r^{-1}(v_2) = \{e_2\}$, $s^{-1}(u_1) = \emptyset$, $r^{-1}(u_1) = \{e_4\}$, $s^{-1}(u_2) = \emptyset$, $r^{-1}(u_2) = \{e_5\}$.

3.1.5. Definition

A vertex u is called infinite emitter if $| (s^{-1}(u) | = \infty$. Moreover, if u is a sink or infinite emitter, then u is a singular vertex otherwise it is called a regular vertex.

3.1.6. Example

[1] Let E be a graph given in the following Figure 3.2, since v has infinite edges, it is infinitely emitter and therefore it is a singular vertex. Also u is a regular vertex because it spreads finite edges. So, the graph E is a finite and non-sequence graph.

E:

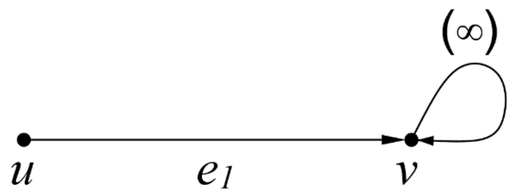


Figure 3.2. Sequence finite graph.

3.1.7. Definition

A path p in a graph E is a sequence of edges $p = e_1 e_2 \ldots. e_n$ such that for i =1, 2, 3, ..., $n - 1$. Also, $s(p) = s(e_1)$ is the source of p and $r(p) = r(e_n)$ is the range of p. If the number of edges forming a path p is infinite, then the path p is called an infinite path. The set of all paths in a graph E is denoted by Path (E).

3.1.8. Definition

Let $p = e_1 e_2 \ldots. e_n$ be a path of length n, if $s(p) = r(p) = u$,, then p is called closed path.A closed simple path based at u is a closed path $p = e_1 e_2 \ldots. e_n$, such that $s(e_i) \neq u$ for every $i > 1$. The set of all closed simple paths based on vertex *u is denoted by* $CSP(u)$. Every closed simple path is a closed path.

3.1.9. Definition

For the path $p = e_1 e_2 .... e_n$, if $s (p) = r (p) = u$ *and* $s^{-1} (e_i) \neq s (e_j)$ for each $i \neq j$, base the vertex u on the path p is called loop. A graph that does not contain a loop is called an acyclic graph, or a noncyclic graph. For the path $p = e_1 e_2 .... e_n$, if there exist*s* $e \in E$ such that $s(e) = s (e_i)$ $(1 \leq i \leq n)$ and $e \neq e_i$, then e is called the output of the p path.

3.1.10. Example

[1] If the n -cornered finite line is generalized for the graph that is M_n graph seen in Figure 3.3:

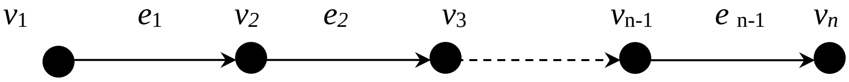


Figure 3.3. n -angular line finite graph.

The vertex set of the graph $M_n^0 = \{v_1, \ldots, v_n\}$, set of edges $M_n^1 = \{e_1 , ... , e_n\}$ and for each e_i (i =1,... n -1), $s (e_i) = v_1$, and $r (e_i) = v_{i+1}$.

3.1.11. Example

[1] According to the rose graph with n leaves, which consists of a single v vertex and n loops represented by R_n as seen in (Figure 3.4), $R_n^0 =\{ v_1 , ..., v_n \}$, $R_n^1 = \{e_1, ..., e_n \}$, and for each e_i (i =1, ..., n -1) $s(e_i) = v_i = r(e_i)$.

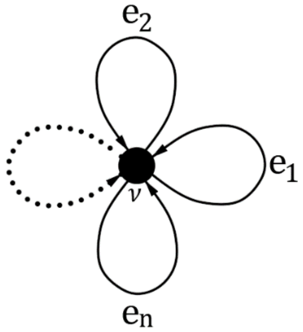


Figure 3.4. n -leaf rose graphy

3.2. PRELIMIMARIES ON LEAVITT PATH ALGEBRA

In the first part of this section, we introduced the Leavitt path algebras and emphasized that it is a ring with local units. In the following, we considered graphs frequently encountered in the literature and defined the properties of the Leavitt path algebras. In the third part, the subject of ideals in Leavitt path algebras briefly introduced and some important results considered. Proofs of some important theorems presented with their references. Throughout the section, the Leavitt path algebra is defined on any directional graph *E*, is denoted by $L_K(E)$ where *K* is any field. Moreover, all the notions are prepared by considering [1].

3.2.1. The Basic Features of Leavitt Path Algebra

3.2.1.1. Definition

Let *R* be a ring, if the isomorphic R_m and R_n free left *R* modules require $m=n$, then *R* is said to have the property of *IBN* (Invariant Base Number).

3.2.1.2. Definition

For a given ring *R* and natural numbers $m<n$, with $R^m \cong R^n$ and $1 < k < m$, if $R^m \ncong R^k$, then the *R* ring has (m,n) type I*BN* property and it is called a ring that does not satisfy the property.

3.2.1.3. Theorem

[1] For each positive integers (m,n) and field *K*, there is one *K*-algebra with $L_K(m,n)$ units. According to the *K*-algebra isomorphism:

(i) The algebra $L_K(m,n)$ has module with type (m,n)

(ii) For any *K*-algebra of the (m,n) module *A*, there is a ϕ: $L_K(m,n) \rightarrow A$, *K*-algebra homomorphism.

3.2.1.4. Definition

[1] Suppose K is any field and that $n > 1$. Then, $(1, n)$ type Leavitt K-algebra is denoted by $L_K(1, n)$ and defined as $K < X_1, X_2, ..., X_n, Y_1, Y_2, ..., Y_n > / < \sum_{i=1}^{n} X_iY_i - 1, X_iY_j - \delta_{ij}1 | 1 \leq i, j \leq n >$

3.2.1.5. Theorem

[1] For each field K, $L_K(1, n)$ is simple where $n > 2$.

3.2.1.6. Definition

A path $\mu = e_1 e_2 .... e_n$ is a finite sequence of edges in E where $s(e_{i+1}) = r(e_i)$ for $1 \leq i \leq n - 1$. Also, $s(\mu) = s(e_1)$ is a source of μ and $r(\mu) = r(e_n)$ is a range of μ. Also, the length of path μ is shown as $n = \ell(\mu)$ or $n = |\mu|$.The vertices angles are treated as paths of zero length. For $v \in E^0$, it is defined as $s(v) = r(v) = v$. For a path $\mu = e_1 e_2 .... e_n$, the set of vertices of path μ is represented as $\mu^0 = \{s(e_i), r(e_i) \mid 1 \leq i \leq n\}$.Moreover, Path($E$) = $\bigcup_{n \geq 0} E^n$ represents all paths set in graph E.

3.2.1.7. Definition

Suppose E is any directional graph and K a field. Also, $(E^1)^* = \{e^* \mid e \in E^1\}$. Accordingly, the Leavitt path algebra on E, the coefficients of which are the elements of K, is a free-joining K-algebra produced by $E^0 \cup E^1 \cup (E^1)^*$, which provides the followings:

(V) $\forall$ v, v' $\in E^0$, v v' = $\delta_{v v'}$ v,

(E1) $\forall$ e $\in E^1$, s(e)e = er(e) = e,

(E2) $\forall$ e $\in E^1$, r(e)e* = e* s(e) = e*,

(CK1) $\forall$ e, e' $\in E^1$, e* e' = $\delta_{e,e'}$ r(e),

(CK2) for all regular vertex $v \in E^0$, $v = \sum_{e \in s^{-1}(v)} ee^*$.

3.2.1.8. Definition

Let E be a graph and A a K-algebra, then, A consists the set $\{a_v | v \in E^0\}$ and the sets $\{a_e | e \in E^1\}$ and $\{b_e | e \in E^1\}$ consist of orthogonal idempotents satisfying the following conditions.

i. For each $e \in E^1$, $a_{s(e)} a_e = a_e = a_e a_{r(e)}$ and also $a_{r(e)} b_e = b_e a_{s(e)} = b_e$,

ii. For each $e, f \in E^1$, $b_f a_e = \delta_{e,f}\ a_{r(e)}$,

iii. For each regular vertex $v \in E^0$, $a_v = \sum_{e \in s^{-1}(v)} a_e b_e$.

Thus, A is called an E-family, and in this case, there is a K-algebraic homomorphism of $L_K(E) \to A$ such that it is $v \to a_v$, $e \to a_e$ *and* $e^* \to b_e$. This is also called the universal property of the Leavitt path algebras.

3.2.1.9. Example [1]

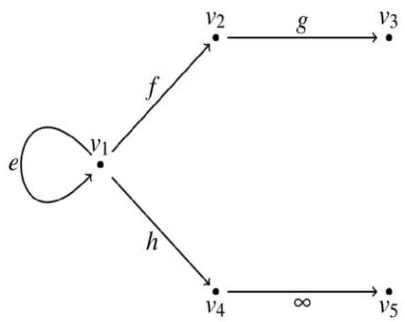


Figure 3.5.

Hence, some operations in $L_K(E)$ are given below:

E1: $v_1 f = f$ and $f = f v_2$.

E2: $v_2 f^* = f^*$ and $f^* = f^* v_1$.

CK1: $f^* f = v_2$, $f^* h = 0 = f^* e$.

CK2: $v_1 = ee^* + f f^* + hh^*$, $v_2 = gg^*$.

Since the vertex v_4 is infinitely radiating and the vertices v_3, v_5 are sink, for these vertices (CK2) is not defined.

3.2.1.10. Definition

For an associative ring R, $F \subseteq R$ consisting of idempotent elements is defined as a set of local units if the following condition is satisfied.

For every $\{r_1, \ldots, r_n\}$ finite subset of R, $fr_if = r_i$, there are $f \in F$, $1 \leq i \leq n$. In other words, for any finite subset N of R, there exists a $f \in F$ such that $N \subseteq f\,R\,f$. For an orthogonal idempotent subsets E of R, if $_RR = \bigoplus_{e \in E} R_e$, then R is said to have sufficient idempotent.

3.2.1.11. Lemma

[1] Suppose E is a graph and K any field. If $\gamma, \lambda, \mu, \rho \in \mathrm{Path}(E)$, then

(i) $(\gamma\lambda^*)(\mu\rho^*) = \begin{cases} \gamma\kappa\rho^* & if\ \mu = \lambda\kappa,\ \kappa \in Path(E) \\ \gamma\sigma^*\rho^* & if\ \lambda = \mu\sigma,\ \sigma \in \mathrm{Path}(E) \\ 0 & \end{cases}$

In accurately, $\lambda = \mu$ if and only if $\ell(\lambda) = \ell(\mu)$ then $\lambda^*\mu \neq 0$. Thus, $\lambda^*\mu = r(\lambda)$.

(ii) The K-effect on $L_K(E)$ is ordinary. It means that $(k\gamma\lambda^*)(k'\mu\rho^*) = kk'(\gamma\lambda^*\mu\rho^*)$ for $k, k' \in K$.

(iii) $L_K(E)$ Leavitt path algebra as a K - vector space $\{\gamma\lambda^* \mid \gamma,\lambda \in Path(E), r(\gamma) = r(\lambda)\}$ consists of mononomials of the form r (γ) = r $(\lambda)\}$. So, every $x \in L_K(E)$ element, $x = \sum_{i=1}^{n} k_i\gamma_i\lambda_i^*$. For $k_i \in K^{\times}$, $\gamma_i, \lambda_i \in Path(E)$, $r(\gamma_i) = r(\lambda_i)$ where $1 \leq i \leq n$.

(iv) $L_K(E)$ is unitary if and only if E^0 is finite. So, $1_{L_K(E)} = \sum_{v \in E^0} v$.

(v) For every $a \in L_K(E)$ there is a $V(a)$ finite set such that $faf = a$, where $f = \sum_{v \in v(a)} v$.

Moreover, $L_K(E)$ is a sufficient locally idempotent ring.

Proof.

(i) By the (CK1), Either $e^* f \in L_K(E)$ is equal to zero or is the vertex $r(e)$ as desired.

(ii) It is follows by the definition of $L_K(E)$.

(iii) It is obtained from (i).

(iv) If E^0 is finite, the proof is clear. Otherwise, there is no unit element in $L_K(E)$.

(v) It is easy to see that sum of the different vertices in $L_K(E)$ is idempotent. On the other hand, for any element $\alpha = \sum_{i=1}^{m} Ki\gamma i\ \lambda i^* \in LK(E)$, V(α) represents the set of vertices on α. Thus, *if* $f=\sum_{v\in V(\alpha)} v$ is defined as $\alpha = f\alpha f$, then we done.

3.2.1.12. Definition

Let E be a graph. Let $\hat{E} = E^1 \cup (E^1)^*$, $u,v \in E^0$ and there is a $\eta = h_1h_2,\ldots,h_m$ such that $s(\eta) = u$, $r(\eta) = v$ and $h_1h_2,\ldots,h_m \in \hat{E}$, then E is called depend. Dependent components of E are $\{E_i\}_{i\in\Lambda}$ graphs. Moreover, $E = \sqcup_{i\in\Lambda}$ can be defined as a discrete combination of connected graphs E_i.

3.2.1.13. Proposition

[1] For a E a graph and field K, the dependent expression of E in terms of its bound components is $E = \sqcup_{i\in\Lambda} E_i$ with $L_K(E) \cong \bigoplus_{i\in\Lambda} L_K(E_i)$.

In the following, we present some Leavitt path algebra examples:

3.2.1.14. Example

[1] R_n represent a graph with only one vertex and n edges.

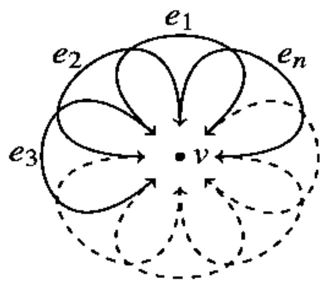


Figure 3.6.

3.2.1.15. Example

[1] [Laurent polynomial ring] The following diagram R_1 with one vertex and edge plays an important role in the theory.

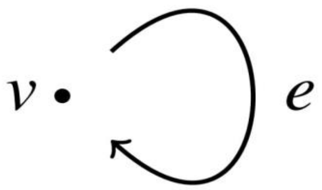


Figure 3.7.

3.2.1.16. Example

[1]. [Matrix algebra] The graph *an* consists *n* vertices and *n*-1 edges.

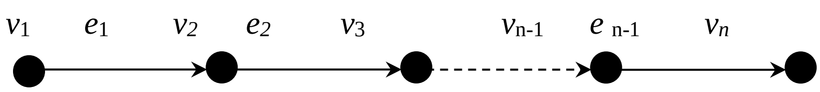


Figure 3.8.

3.2.1.17. Proposition

[1] If K is a field, and $n \geq 2$ where n is any positive integer, then $L_K(1, n) \cong LK(R_n)$.

3.2.1.18. Definition

[1] Let K be a field, the Laurent polynomial is a K-algebra obtained by x and y that provides a relation $xy = yx = 1$ and is denoted by $K[x, x^{-1}]$.

3.2.1.19. Proposition

[1] If K is a field, then we have $L_K(R_1)$. $\cong K[x, x^{-1}]$.

Proof. Through the (CK_1), we have $e^*e = v = 1$ in $L_K(R_1)$. Also, since e is just edge of v, $ee^* = 1$. is obtained from (CK_2) in $L_K(R_1)$.

3.2.1.20. Theorem

[1] $M_n(K) \cong L_K(A_n)$ where K is a field and $n \geq 1$ is any positive integer.

Proof. Suppose $\{f_{i,j} : 1 \leq i, j \leq n\}$ represents the matrix units in $M_n(K)$.Accordingly, if the transform $\varphi: LK(A_n) \rightarrow M_n(K)$ is defined as $\varphi(v_i) = fi,i$, $\varphi(e_i) = f_{i,i+1}$, it can be easily shown that φ is a K-algebraic isomorphism.

3.2.1.21. Example

[1] The graph below is called the Toeplitz graph and is denoted by E_T.

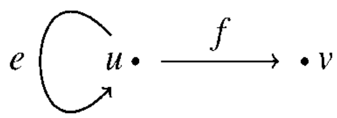


Figure 3.9.

3.2.2. Ideals of the Leavitt Path Algebras

3.2.2.1. Definition

Let R be a ring or algebra. The ideal of R obtained by X, with $X \subseteq R$, is denoted by I (X).

3.2.2.2. Definition

For a graph $E = (E^0, E^1, r, s)$, we have the followings:

- Let $\mu = e_1e_2...e_n \in Path(E)$ and $\ell(\mu) \geq 1$, if $s(\mu) = v = r(\mu)$, then μ is called a v-based closed path.
- Let $\mu = e_1e_2...e_n$ be a closed path that depends on v. If $s(e_i) \neq v, i \geq 2$, it is called a v-dependent simple closed path based. A set of simple closed paths based on v in E is denoted by $CSP(v)$.
- Let $\mu = e_1e_2...e_n \in CSP(v)$. If $s(e_i) \neq s(e_j)$ for each $i \neq j$, then μ is called a v-based loop.
- A loop of length 1 is called curl.

3.2.2.3. Definition

Let E be a graph and $v \in E$, if $|CSP(v)| = 0$ or $|CSP(v)| \geq 2$, then E satisfy the K-condition.

3.2.2.4. Definition

For a graph $E = (E^0, E^1, r, s)$, the preorder, $\geq$, is defined on E^0 as follows:

$P \geq v$ if and only if there is a path $\mu \in Path(E)$ where $s(u) = p$, $r(u) = v$.

3.2.2.5. Definition

Let E be a graph and $v \in E^0$. The set $T(v) = \{w | w \in E^0, v \geq w\}$ is called tree of v. For a subset $X \subseteq E^0$, it is defined as $T(X) = \cup_{v \in \mathrm{x}} T(v)$.

3.2.2.6. Definition

Let E be a graph and $H \subseteq E^0$.

- If $v \in H$ and $w \in E^0$ such that $v \geq w$ requires $w \in H$, then H is said to be inherited.
- If $r(s^{-1}(v)) \subseteq H$ with $v \in Reg(E)$ such that $v \in H$ then H is called saturated.

3.2.2.7. Definition

Let E be any graph and $X \subseteq E^0$. The smallest heritable-saturated subset of E^0 containing the set X is called the hereditary-saturated closure of X and is denoted by $\bar{X}$.

3.2.2.8. Lemma

[1] For a graph E, the hereditary-saturated closure of X, with $X \subseteq E^0$, is $\bar{X}$ = $\cup_{n=0}^{\infty} \Lambda_n(X)$ where

- $\Lambda_0(X) = T(X) = \{v \in E^0 \, | x \geq v, \exists x \in X\}$ and
- $\Lambda_n(X) = T(X) = \{y \in E^0 \colon 0 < |s^{-1}(y)| < \infty, \; r(s^{-1}(y)| \subseteq \Lambda_{n-1}(X)\} \cup \Lambda_{n-1}(X)$, $n \geq 1$.

Proof. It is clear that any heritable-saturated subset of set E^0 containing X also contains the set $\sum_{n \geq 0} Xn$. Also, since every X_n set is inherited, the set $\sum_{n \geq 0} X_n$ is also inherited. Now we need to prove that $\sum_{n \geq 0} X_n$ is saturated. Let $v \in Reg(E)$ be a vertex so that $r(s^{-1}(v)) \subseteq \sum_{n \geq 0} X_n$. Since $X_n \subseteq X_{n+1}$ and $r(s^{-1}(v))$ is finite, there is $N \in \mathbb{N}$, so $r(s^{-1}(v)) \subseteq X_n$. Therefore, $v \subseteq X_{n+1}$ as desired.

PART 4

INVERSE SEMIGROUPS

In this chapter, we focused on studying on inverse semigroups. This subject was firstly introduced by Wagner in 1952 [3] and independently by Preston in 1954 [4]. In the following years, numerous studies on this subject continued. For more details on inverse semigroups, we refer readers to see [5, 6, 7]. The aim of this chapter is to give properties of inverse semigroups and study on its ideals. The main reference for this chapter is [2].

4.1. PARTIAL BIJECTIONS

Let A and B be two sets, and f a function from A to B. If f is defined from a subset of A to a subset of B, then f is called partial function. The subset of A consisting of all elements a ∈ A is called as a domain of f and denoted by dom$f$. im$f$ = f(dom$f$) is a image of f which is also a subset of B.

Let A and B be two sets. 0_{BA} is a unique empty partial function defined from A to B. Empty function refers to any partial function of this type.

1_X is an identity function on $X \subseteq$A. It is also a partial function from A to A. These partial functions are called partial identites. d(f) is the partial identity function on dom$f$ and r(f). is the partial identity function on im$f$. The identity function 1_A on A and the identity function 1_0 on the empty subset of A that is the empty function from A to itself. To shortly, we define these functions by 1 and 0 respectively.

4.1.1. Definition

Let g be a partial function defined from A to B and f a partial function defined from B to C, their composite is also a partial function from A to C and denoted by *fog*, where the domain of fog is dom($f$og) = g^{-1} (dom$f$ ∩ img) and if a ∈ dom (f o g) then (f o g) (a) = f(g(a)). The image of fog is denoted by f (dom$f$∩ *img)*. If dom$f$ and img have empty intersection, then fog is the empty function. Also, may we generally write be *fg* as short for *fog*.

4.1.2. Definition

If f is a partial bijection from A to B, the empty functions and all partial identities are partial bijections. Then the inverse of f is the partial bijection from B to A, denoted by f^{-1}. Thus, the domain of f^{-1} is im$f$ and be the image is dom$f$. The formation of partial bijections is also a partial bijection.

4.1.3. Proposition

Let A, B and C be sets, and f: A → B a partial bijection. Then we have the followings:

(i) $ff^{-1}=1_{\mathrm{dom}f}$ is a partial identity on A, and $ff^{-1}=1_{\mathrm{im}f}$ is a partial identity on B.

(ii) Let g:B→A be a partial bijection, then one gets $g=f^{-1}$ if and only if $f=fgf$ and $g=gfg$.

(iii) $(f^{-1})^{-1} = f$.

(iv) For all partial identities 1_Y and 1_X, we have $1_X 1_Y=1_{X\cap Y} = 1_Y 1_X$ where X, Y ⊆ A.

(v) For any partial bijection g: B→C, we have $(gf)^{-1} = f^{-1} g^{-1}$.

Proof:

(i) Since $x \in \mathrm{dom}f$, we have $x \in \mathrm{dom}(f^{-1}f)$. So, one gets that domain of f is the same as $f^{-1}f$. However, $f^{-1}f$ is the identity function on its domain. Thus, $f^{-1}f = 1_{\mathrm{dom}f}$.

(ii) By the hypothesis we have $g: B \to A$. Also let $f = fgf$ and $g = gfg$, $y \in \mathrm{dom}g$ and $x = g(y)$. So, $x = (gfg)(y)$ thus $x = g(f(x))$. Since g is a partial bijection, $g \subseteq f^{-1}$. Let $y \in \mathrm{dom}(f^{-1})$ and $x = f^{-1}(y)$ so $f(x) = y$ and if we substitute in the equation (i), we have $(fgf)(x) = y$ this means that $f(g(f(x)) = y$ thus $f(g(y)) = y$. Now let $x = g(y)$, thus $f(x) = y$ then f is a partial bijection which gives that $f^{-1} \subseteq g$. So $g \subseteq f^{-1} = f^{-1} \subseteq g$. Hence $f^{-1} = g$ as desired.

(iii) We have $f = fgf$ and $g = f^{-1}$ so $f = fgf$. Thus, $f^{-1} = (fgf)^{-1} = f^{-1}g^{-1}f^{-1} = (f^{-1}g^{-1})f^{-1} = (f^{-1}g^{-1})g$. So one gets that $(f^{-1})^{-1} = ((f^{-1}g^{-1})g)^{-1} = g^{-1}gf = f$ as desired.

(iv) Since $1_x1_y = 1_{x\cap y} = 1_{y\cap x}$, $x \cap y = y \cap x$. Let $a \in \mathrm{dom}(1_x1_y)$ then $(1_x1_y)(a) = 1_x(1_y(a)) = a$. Hence 1_x1_y and 1_y1_x are a partial identities, therefore $\mathrm{dom}(1_x1_y) = x \cap y$ and $\mathrm{dom}(1_y1_x) = y \cap x$ this means that $1_x1_y = 1_{x\cap y} = 1_{y\cap x}$.

(v) Since partial identities commute, we have

$gf(f^{-1}g^{-1})gf = g(ff^{-1})(g^{-1}g)f = g(g^{-1}g)(ff^{-1})f = gf$.

Moreover, $(f^{-1}g^{-1})gf(f^{-1}g^{-1}) = f^{-1}g^{-1}$ implies $(gf)^{-1} = f^{-1}g^{-1}$.

4.1.4. Note

Each element in a semigroup equal to its square is called an idempotent.

4.2. INVERSE SEMIGROUPS

The inverse semigroup S is defined by Wagner and Preston by the followings:

i. S is regular i.e., for each element $x \in$ S there is another element y, called an inverse of x, such that $x = xyx$ and $y = yxy$.

ii. The idempotents elements of the S commute.

4.2.1. Proposition

A regular semigruop is inverse if and only if its idempotetns commute.

Proof Let S be a regular semigroup and its idempotents commute and u and v inverse of x. Then we have u = uxu = u(xvx)u = (ux)(vx)u, v = vxv and u =uxu. Since ux and vx are idempotents and commute, one gets u = (vx)(ux)u =v(xux)u= vxu = (vxv)xu = v(xv)(xu). Since ux and vx are idempotents commute, we have that u = v(xu)(xv) = v(xux)v = vxv = v.This means u = v. For the converse, it is easy to see that the result multiplication of two idempotents f, e has an idempotent inverse in a regular semigroup. Because, let x = $(\mathrm{e}f)'$ is inverse for ef. Thus, the element $f$xe is an idempotent inverse of ef.

First, suppose that S is a semigroup, and each element has a single inverse. Secondly, we must prove that any two idempotents elements have a fixed inverse. Such as ef = fe. From the above we have $f(\mathrm{e}f)'\mathrm{e}$ is a fixed inverse of ef. Through the uniqueness of inverses, we find $(\mathrm{e}f)' = f(\mathrm{e}f)'\mathrm{e}$ is idempotent element, all idempotent elements are self-inverse. We also find that the inverse of ef is $(\mathrm{e}f)'$. Through, the uniqueness of inverses, we find $\mathrm{e}f = (\mathrm{e}f)'$. This means that e$f$ is idempotent. Since the multiplication operation commutative, $\mathrm{e}f(f\mathrm{e})\mathrm{e}f = (\mathrm{e}f)(\mathrm{e}f) = \mathrm{e}f$, and $f\mathrm{e}(\mathrm{e}f)f\mathrm{e} = f\mathrm{e}$. It results that fe idempotent. This means that fe and ef, they are inverses for ef. Thus $\mathrm{e}f = f\mathrm{e}$.

In an inverse semigroups, if any element s$\in$ S,then there is another single element $s^{-1} \in$ S, where $s^{-1} = s^{-1}s\, s^{-1}$ and $s = s\, s^{-1}s$ is called the a inverse of S and defined by s^{-1}.

4.3. ORDERED GROUPOIDS

Let g be a partial function from A to B and f a partial function from B to C. We say that f. g is bound product of f, g, if and only if dom f= im g and is denoted by $f\cdot$ g $=f$ g. The bound product f. g is fully defined if d (f) = r (g).

If f and g are partial functions defined from A to B, then dom$f$ $\subseteq$domg and f(x) = g(x) for every x $\in$ dom f, then is $f \subseteq$g thus f is a restriction of g.

4.3.1. Proposition

Let A, B and C be sets and f, g: A→B partial bijections, then

(i) If $f \subseteq g$ then $f^{-1} \subseteq g^{-1}$.

(ii) Let p, q: B→C be partial bijections. If $f \subseteq g$ and $p \subseteq q$, then $pf \subseteq qg$.

(iii) $f \subseteq g$ precisely if there exists a partial identity $l_N \in I(A)$ such that $f = g\, l_N$.

Proof:

(i) Let f, g: A→B, and $f \subseteq g$ i.e., dom $f \subseteq$dom g also $\mathrm{im}(f^{-1}) = \mathrm{dom}(f)$ and $\mathrm{dom}(f^{-1}) = \mathrm{im} f$. So $f \subseteq g$, dom $f \leq$ domg, $\mathrm{im}(f^{-1}) = \mathrm{dom} f \leq \mathrm{dom} g = \mathrm{im}(g^{-1})$. This means that $\mathrm{im}(f^{-1}) = \mathrm{im}(g^{-1})$. Hence $f^{-1} \subseteq g^{-1}$.

(ii) Let p, q: B→C. and $pf \subseteq qg$ i.e., dom $pf \subseteq$dom qg also $\mathrm{im}(p^{-1}f^{-1}) = \mathrm{dom}(pf)$ and $\mathrm{dom}(p^{-1}f^{-1}) = \mathrm{im}\, pf$. So $pf \subseteq qg$, dom $pf \leq$ dom qg, $\mathrm{im}(p^{-1}f^{-1}) = \mathrm{dom}\, pf \leq \mathrm{dom}\, qg = \mathrm{im}(q^{-1}g^{-1})$. This means that $\mathrm{im}(p^{-1}f^{-1}) = \mathrm{im}(q^{-1}g^{-1})$. Hence $p^{-1}f^{-1} \subseteq q^{-1}g^{-1}$.

(iii) Let $f \subseteq$ g and Let $l_N = f^{-1}f$ where $\mathbb{N}$ = dom f thus $f \subseteq g\, l_N$, $f \subseteq \mathrm{dom}(g\, l_N)$. Let $x \in \mathrm{dom}(g\, l_N)$. Then l_N (x) is defined as $x \in \mathrm{dom} f$. Thus $\mathrm{dom} f = \mathrm{dom}(g\, l_N)$. Hence $f = g\, l_N$.

Conversely, assume that $f = g\, l_N$ for a partial identity l_N. Let $x \in \mathrm{dom}(f)$, then f(x) and so $(g\, l_N)(x)$ are defined. Moreover $x \in \mathrm{dom}(f)$. Thus dom$f \subseteq$dom g. But we have that $(g\, l_N)(x) = g(x)$. Thus $f \subseteq g$.

If f is a partial function from A to B and $X \subseteq \text{dom } f$. We say that $(f \mid X)$ is a new partial function the restriction of f to X, there is also the partial function, from A to B be dom $(f \mid X) = X$ such as $(f \mid X)(x) = f(x)$for each $x \in X$. If Y is a subset of im$f$. We say that $(Y \mid f)$, the corestriction of f to Y, to be the partial function from A to Y with dom $(Y \mid f) = f^{-1}(Y)$ such as $(Y \mid f)(x) = f(x)$for each $x \in$ dom $(Y \mid f)$.

4.3.2. Proposition

Let $f, g \in I(X)$ and $B = \text{dom} f \cap \text{im} g$. Then $fg = (f \mid B) \cdot (B \mid g)$.

Proof.

By the definition, dom $((f \mid B) \cdot (B \mid g))$ = dom $((B \mid g))$, and dom $((B \mid g)) = g^{-1}(B) = g^{-1}(\text{dom } f \cap \text{im } g)$. Hence dom $((f \mid B) \cdot (B \mid g)) = \text{dom}(fg)$.

4.4. INVERSES PROPERTIES

4.4.1. Proposition

Let S be an inverse semigroup, then we have the followings:

(i) For every $s \in S$, both ss^{-1} and $s^{-1}s$ are idempotents and $(ss^{-1})s = s$ and $s(s^{-1}s) = s$.
(ii) $(s^{-1})^{-1} = s$ for every $s \in S$.
(iii) For every idempotent e in S and any $s \in S$, $s^{-1}es$ is idempotent.
(iv) For every $e^{-1} = e$. If and only if e be idempotent in S.
(v) $(s_1 \ldots s_n)^{-1} = s_n^{-1} \ldots s_1^{-1}$ for each $s_1, \ldots, s_n \in S$ if $n \geq 2$.

Proof.

(i) Let $s = ss^{-1}s$, $s^{-1} = s^{-1}ss^{-1}$. Then we have $(s^{-1}s)^2 = s^{-1}(ss^{-1}s) = s^{-1}s$ and $(ss^{-1})^2 = s(s^{-1}ss^{-1}) = ss^{-1}$. Hence $s^{-1}s$ and ss^{-1} are idempotents.

(ii) We have that $s^{-1} = s^{-1}x\ s^{-1}$ and $x = x\ s^{-1}x$, let $e = s\ s^{-1}$, then by the uniqueness of inverses result follows.

(iii) Let $(s^{-1}e\ s)^2 = s^{-1}e\ s\ s^{-1}e\ s$. Since e and s are idempotents commute, so we have $s^{-1}e\ s\ s^{-1}e\ s = (s^{-1}s\ s^{-1})\ e\ (e\ s) = s^{-1}\ e\ s$, as desired.

(iv) Let e^{-1} be an idempotent, so $(e^{-1})^2 = e^{-1}\ e^{-1} = e$, this means that $e^{-1} = e$.

(v) When $n = 2$, we find $(s_1\ s_2)^{-1} = s_2^{-1}s_1^{-1}$; $s_1 ,s_2 \in S$. when $n > 2$ This case is a generalization of the two cases (i) - (ii) and prove holds by the induction method. We'll write $d\ (s) = s^{-1}s$, $r\ (s) = ss^{-1}$.

By the above, we find deduce the characteristics:

(i) $(s^{-1})^{-1} = s$ for all $s \in S$.

(ii) $(s\ t)^{-1} = t^{-1}\ s^{-1}$ for all $s, t \in S$.

4.4.2. Proposition

Let S be an inverse semigroup. Then we have

(i) For any idempotent e and s, there is an idempotent f such as, $es = sf$.

(ii) For any idempotent e and s, there is an idempotent f such as, $se = fs$.

Proof.

(i) Let $f = s^{-1}\ es$ be an idempotent, then $sf = s(s^{-1}es) = (ss^{-1})\ es = e(ss^{-1})\ s = es$.

(ii) Let $f = s^{-1}\ es$ be an idempotent, then $fs = (s^{-1}es)\ s = (s^{-1}s)\ es = (s\ s^{-1}\ s)\ e = se$.

4.4.3. Proposition

Let A be a non-empty subset of S, the product of the intersection of all the inverse subsets of S that contain A is called an inverse subset of S. It consists carefully of all products of elements drawn from the set $A \cup A^{-1}$.

4.4.4. Proposition

Groups are exactly the inverse semigroups with precisely one idempotent.

Proof.

Let S be an inverse semigroup with precisely one idempotent, say e. Then $s^{-1}s = e = s\ s^{-1}$ for each $s \in S$. But es = $(s\ s^{-1})\ s = s = s\ (s\ ^{-1}s)$ = se, and so e is the identity of S. Hence S is a group.

Suppose S is an inverse semigroup and has an identity we say that S is an inverse monoid for example an element; e and symbolize it S^1. If S has a zero element, then S is an inverse semigroup with zero and denoted by S^o. Each inverse semigroup can be converted to an inverse monoid or inverse semigroup with zero through aligning zero or adjoining an identity by the following way.

4.4.5. Example

Let S is a semigroup. We define a monoid S^1 in the following. If S is inverse, then S^l is an inverse monoid S^l. If S is a monoid, then $S^l = S$. If S isn't a monoid, then $S^l= S \cup \{1\}$ also with the multiplication in S extended to S^l by defining 1s = s1= s for each $s \in S$ and 1.1 = 1. Then S^l is a monoid.

4.4.6. Example

Let S is a semigroup. We define a zero monoid S^o in the following. if S is inverse then S^o is an inverse zero S^o. If S is a zero monoid, then $S^o= S$. If S isn't a zero, then $S^o= S \cup \{1\}$ also with the multiplication in S extended to S^o by defining 0s = s0= s for each $s \in S$ and 0.0 = 0. Then S^o is a zero.

4.5. IDEALS

Suppose X and Y are two subgroups of a semigroup S. Where X ={x} and Y = {y}, their product XY is a subset of the product of X elements with Y elements, denoted by xY instead of {x}Y and the same for the other set Xy by X {y}.

Suppose S is a semigroup, a subset A of S, If sa $\in$A where a $\in$A and s $\in$ S are called left ideals, and if as $\in$ A are called right ideals, where a $\in$A and s $\in$ S. Each of the subsets representing left and right ideals is called ideals, then is considered the result of the intersection of any empty set of right ideals is a right ideal and similarly for the left ideals (left ideal). We say about the left ideal that contains s the basic left ideal, if there is a smaller left ideal that contains s where s $\in$ S. And the same for the right ideal.

Suppose S is arbitrary group. We say about the basic left ideal that contains s is a generator of this ideal, and defined by the S^1s.Likewise, we say about the basic right ideal that contains s is a generator of this ideal and defined by the s S^1.

Suppose S is a semigroup, if s = $(ss^{-1})s = s(s^{-1}s)$, where s$\in$ sS and s$\in$ Ss. then We say about the right ideal that contains s is called the basic right ideal and symbolizes by sS. We say about the left ideal that contains s is called the basic left ideal and symbolizes by Ss. And we say both basic right ideal and basic left ideal is called the basic two-sided ideal and symbolizes by SsS.

Suppose A is a group, for every a$\in$ A, then aA = Aa = A. We note that ideals have no influence in group theory, on the contrary, in inverse semigroup theory they have a very important role.

4.5.1. Proposition

Let S is an inverse semigroup. Then

(i) if $aS = aa^{-1}S$ for each $a \in S$, there exists aa^{-1} is the only idempotent generator for aS.

(ii) if $Sa = Sa^{-1}a$ for each $a \in S$, there exists $a^{-1}a$ is the only idempotent generator for Sa.

(iii) for e and f are idempotents, so $eS \cap fS = efS$.

(iv) for e and f are idempotents, so $Se \cap Sf = Sef$.

Proof.

(i) We have that $aS = aa^{-1}aS = (aa^{-1})\, aS \subseteq (aa^{-1})\, S \subseteq aS$, so $aS = aa^{-1}S$. Now let e any idempotent like that $aS = eS$, this means that $aS = aa^{-1}S = es$, so $aa^{-1} = es$ and $e = aa^{-1}t$ for $s, t \in S$. Since aa^{-1} and e idempotents commute, $eaa^{-1} = aa^{-1}$ and $aa^{-1}e = e$. Hence $aa^{-1} = e$, so $aS = aa^{-1}S$.

(ii) We have that $Sa = Saa^{-1}a = Sa(a^{-1}a) \subseteq S(a^{-1}a) \subseteq Sa$, so $Sa = Sa^{-1}a$. Now let e be any idempotent such that $Sa = Se$, this means $Sa = Sa^{-1}a = se$. Thus $a^{-1}a = se$ and $e = a^{-1}a\, t$ for $s,t \in S$. Since $a^{-1}a$ and e idempotents commute, $ea^{-1}a = a^{-1}a$ and $a^{-1}ae = e$. Hence $a^{-1}a = e$, so $Sa = Sa^{-1}a$.

(iii) Suppose that $a \in eS \cap f S$, then $ea = a$ and $fa = a$. Thus $(ef)a = e(fa) = ea = a$, and so $a \in efS$. To prove the Converse, if $a \in efS$ then $ea = a$ and $fa = a$ since and f are idempotents commute. So, $a \in eS \cap fS$, Thus, $eS \cap fS = efS$.

(iv) Let $a \in Se \cap Sf$, then $ae = a$ and $af = a$. Thus $a(ef) = (af)e = ae = a$, and so $a \in Sef$. Conversely, if $a \in Sef$ then $ae = a$ and $af = a$ since e and f are idempotents commute. So, $a \in Se \cap Sf$, Thus, $Se \cap Sf = Sef$.

4.6. THE NATURAL PARTIAL ORDER

The symmetric inverse monoid is ordered by the restriction order. One can define a relation $\leq$ for some idempotent e for all inverse semigroup S, in the following way:

$s \leq t \leftrightarrow s = te.$

4.6.1. Proposition

Suppose S is an inverse semigroup. Then the followings hold:

(i) $s \leq t$.

(ii) $s = ft$ for some idempotent f.

(iii) $s^{-1} \leq t^{-1}$.

(iv) $s = ss^{-1}t$.

(v) $s = ts^{-1}s$.

Proof.

(i) Suppose $s = te$, then $s = ft$. Let $f = t^{-1}et$ be an idempotent, then $S=(t^{-1}e\ t)t=(t\ t^{-1})(te) =(t\ t^{-1}t)e= te$. Thus $s= te$ and $s= fe$ and so $s \leq t$.

(ii) Suppose $s =ft$ for idempotent f. Thus, $s^{-1}=t^{-1} f^{-1}$, by definition 3.6. We find that $s^{-1}\leq t^{-1}$.

(iii) Suppose $s^{-1} \leq t^{-1}$ for some idempotent e. Then $s^{-1} = t^{-1}e$. But $s = et$, $es = s$ and so $e\, s\, s^{-1} = s\, s^{-1}$. Thus $s = ss^{-1}t$.

(iv) Suppose $s^{-1} =ss^{-1} t$ for some idempotent e. Then $s=te$. But $se= s$ and $s^{-1}se=s^{-1}s$. Thus $s = ss^{-1}t$.

(v) It holds by (iv).

Suppose $(K, \leq)$ is a partiallly ordered set. So, we have a subset H of K be an order ideal if $d \leq z \in H$ denotes that $d \in H$. The principal order ideal of K containing d is the set $[d] = \{z \in K: z \leq d\}$ Moreover, if M is any subset of K then $[M] = \{z \in K: z \leq a \text{ for some } a \in M\}$. The order ideal is generated by M.

4.6.2. Proposition

Let S be an inverse semigroup. Then

(i) The relation $\leq$ is a partial order on S.

(ii) For idempotents $e, f \in S$ we have $e \leq f$ if and only if $e = ef = fe$.

(iii) If $s \leq t$ and $x \leq y$, then $sx \leq ty$.

(iv) If $s \leq t$, then $ss^{-1} \leq tt^{-1}$ and $s^{-1}s \leq t^{-1}t$.

(v) E(S) is an order ideal of S.

Proof.

(i) We have that $s = s(s^{-1}s)$, this means that the relationship is reflexive. Also suppose that $s \leq t$ and $t \leq s$. Thus $s = ts^{-1}s$ and $t = st^{-1}t$. It means that $s = ts^{-1}s = st^{-1}ts^{-1}s = st^{-1}t = t$, so that a relation is antisymmetric. Let $s \leq t$ and $t \leq v$, so $s = te$ and $t = vf$ for some idempotetns e, f. Thus $s = te = (vf)e = v(fe)$. This means that $s \leq v$.

(ii) Let $e \leq f$. So, $e = fi$. For idempotent i, $fe = e$ and thus the $e = fe = ef$. To prove the opposite, suppose S be an inverse semigrup and e, $f \in S$, then $e \leq f$ ana thus that e, f idempotents.

(iii) Assume that $s \leq t$ and $x \leq oy$. For some idempotents e, f. Like that $s = te$ and $x = yf$. Thus $sx = teyf$. By the Proposition 3.4.2, we have $ey = yi$ for some idempotent i. Hence $sx = ty(if)$, and so $sx \leq ty$.

(iv) By Proposition 3.6.1. Let $s \leq t$, then $s^{-1}=t^{-1}f$. If $s^{-1}\leq t^{-1}$, then $s^{-1}s=t^{-1}tf$. If $s^{-1}s \leq t^{-1}t$. We also have $ss^{-1} = f\ t\ t^{-1}$ for idempotent f. So $ss^{-1}=tt^{-1}f$. Thus, by the definition $ss^{-1} \leq tt^{-1}$. Hence $s^{-1}s \leq t^{-1}t$ and $ss^{-1} \leq tt^{-1}$.

(v) E(S) is closed under multiplication. By the definition of the natural partial order, E(S) is called the natural partial order on S.

Suppose S is a semigroup and $\leq$ partial order defined in S. If for each x, y, z and d S. So, we have $x \leq y$ and $z \leq d$ which means that $xz \leq yd$. We say S be compatible with multiplication and partially ordered by $\leq$. Note that if G is an inverse subset of S, the natural partial order is defined on G agrees with the restriction on G for the natural partial order on S.

suppose (K, $\leq$) be positive. We say κ be the lower bound of a and b if $\kappa \leq \upsilon,\mu$.We say κ be the greatest lower bound if κ is the largest lower bound and denoted by $\upsilon \wedge \mu$. In a poset, if every pair of elements has a greatest lower bound, then it is called meet semilattice.

4.6.3. Proposition

Let S be any semigroup. A relation $\leq$ on E(S) defined by
$e \leq f \leftrightarrow e = ef = fe$.
Thus, $\leq$ is a partial order on E(S). Moreover, if S is an inverse semigroup, then (E(S), $\leq$) is a meet semilattice.

Proof.

(i) Reflexivity: suppose $e \in E(S)$, thus $e \leq e$ since e is idempotent.
(ii) Antisymmetric: Let $e \leq f$ and $f \leq e$. Then $e = ef = fe$ and $f = fe = ef$. Thus $e = f$.
(iii) Transitivity: Let $e \leq f$ and $f \leq g$. Then $e = ef = fe$ and $f = fg = gf$. This leads $e = ef = fe$, $e = ef = e(fg) = (ef)g = eg$. Also, $f = fg = gf$, $e = fe = (fg)e = ge$. It is produced from both $e = ge = eg$. Thus $e \leq g$. Now let S is an inverse semigroup. Let $e, f \in E(S)$. Then $(ef)e = (fe)e = fe$. This leads to $ef \leq e$. Also $(ef)f = (fe)f = ef = f(ef) = fe$. This leads to Thus $ef \leq f$. Also, suppose $i \leq e, f$. Then $i(ef) = (ie)f = if$ as $i \leq e$, and $if = i$ as $i \leq f$. So, $i \leq ef$. this results in $e \wedge f = ef$. This means that (E(S), $\leq$) is a meet semilattice.

4.6.4. Proposition

All meet semilattices are inverse semigroups, and also an inverse in which every element is an idempotent is a meet semilattice.

Proof.

Suppose (G, $\leq$) be a meet semilattice. So, one gets $e = e \wedge e$ for all element $e \in$ G. This means that (G, $\wedge$) be an inverse semigroup where any element is idempotent. To prove the converse clear by Proposition 3.6.3. Suppose S is semigroup and define a relation $\leq$ on G by

$$e \leq f \leftrightarrow e = ef = fe.$$

Now suppose $i \le e,\ f$. Thus, $i(ef) = (ie)f = if$ as $i \le e$, and $if = i$ as $i \le f$. So, $i \le ef$. this results in $e \wedge f = ef$. This means that (G, $\le$) is a meet semilattice.

4.6.5. Proposition

An inverse semigroup is a group if and only if natural partial order is the equality relation.

Proof.

Let the natural partial order is the equality relation. And the natural partial order has two idempotents e, f then $ef \le e,\ f$. So, $e = f$. We know S has single only idempotent. This means S an inverse semigroup. The converse is immediate by Proposition 3.4.4. Let S be an inverse semigroup, and it S has completely one idempotent, so $s^{-1}s = e = ss^{-1}$ for $s \in S$ but $es = (ss^{-1})\, s = ss^{-1}s = s = s(s^{-1}s) = se$. Thus, e is the identiity of S. Hence S is group.

4.7. THE COMPATIBILITY RELATIONS

If f, g $\in$ I(X), then $f \cup g$ is a partial function accurately, then $f\, g^{-1}$ is an idempotent and if $f \cup g$ is a partial bijection accurately, then $f\, g^{-1}$ and $f^{-1}g$ are idempotents. For s, t $\in$ S, the left compatibility relation is

$$s \sim_l t \leftrightarrow s\, t^{-1} \in E(S),$$

the right compatibility relation is

$$s \sim_r t \leftrightarrow s^{-1}\, t \in E(S),$$

and intersection of these two relations is

$$s \sim t \leftrightarrow s\, t^{-1},\ s^{-1}\, t \in E(S).$$

All previous relations are symmetric, reflexive and but that any of them have don't to be transitive.

4.7.1. Proposition

Let S be an inverse semigroup and s, t $\in$ S. Then

(i) $s\sim_{l}$ t if and only if the greatest lower bound $s \wedge t$ of s, t exists and $(s \wedge t)^{-1}(s \wedge t)=s^{-1}st^{-1}t$.

(ii) $s \sim_{r} t$ if and only if the greatest lower bound $s \wedge t$ of s, t exists and $(s \wedge t)(s\wedge t)^{-1}=ss^{-1}tt^{-1}$.

(iii) $s \sim t$ if and only if the greatest lower bound $s \wedge t$ of s, t exists and $(s \wedge t)(s \wedge t)^{-1}=ss^{-1}tt^{-1}$ and $(s \wedge t)^{-1}(s \wedge t)=s^{-1}st^{-1}t$.

Proof.

(i) Let $s\sim_{l} t$ and say $x = st^{-1}t$, then $x \le s$ and $x \le t$ since st^{-1} is idempotent. Now let $y \le s, t$, then $y \le s$ and $y \le t$, so we have $y^{-1}y \le t^{-1}t$ which implies that $y \le st^{-1}t = x$. Thus, $x = s \wedge t$. Also $x^{-1}x = (st^{-1}t)^{-1}(st^{-1}t) = (t^{-1}ts^{-1})(st^{-1}t) = s^{-1}st^{-1}t$.
Conversely, suppose that $s \wedge t$ exists and $(s \wedge t)^{-1}(s \wedge t) =s^{-1}st^{-1}t$. Put $x =s \wedge t$. Then $x = sx^{-1}x$ and $x = tx^{-1}x$. Thus, $sx^{-1}x = tx^{-1}x$, and so $st^{-1}t = ts^{-1}s$. Thus, $st^{-1} = ts^{-1}st^{-1}$ which is idempotent. This means that, $s\sim_{l} t$.

(ii) Let $s\sim_{r} t$ and say $a = ss^{-1}t$, then $a \le s$ and $a \le t$ since $s^{-1}t$ is idempotent. Now let $b \le s, t$, then $b \le s$ and $b \le t$ so that $bb^{-1} \le tt^{-1}$ and this implies that $b \le ss^{-1}t = a$. Hence $a = s \wedge t$. Also $aa^{-1} = (ss^{-1}t)(ss^{-1}t)^{-1} = (ss^{-1}t)(t^{-1}ss^{-1}) = ss^{-1}tt^{-1}$.
On the other hand, suppose that $s \wedge t$ exists and $(s \wedge t)(s \wedge t)^{-1} =s^{-1}st^{-1}t$. Put $a =s \wedge t$. Then $a = saa^{-1}$ and $a = taa^{-1}$. Thus $saa^{-1} = taa^{-1}$, and so $stt^{-1} = tss^{-1}$. Hence $st^{-1} = tss^{-1}t^{-1} = ss^{-1}tt^{-1}$ it is idempotent. Then $s\sim_{r} t$.

(iii) It follows by (i) and (ii).

4.7.2. Proposition

Let S be an inverse semigroup. Then

(i) If $s \sim_{\iota} t$, then we have $s \wedge t = st^{-1}t = ts^{-1}t = ts^{-1}s = st^{-1}s$.

(ii) If $s \sim_r t$, then we have $s \wedge t = ss^{-1}t = st^{-1}s = tt^{-1}s = ts^{-1}t$.

(iii) If $s \sim t$, then we have $s \wedge t = st^{-1}t = ts^{-1}t = ts^{-1}s = st^{-1}s = ss^{-1}t = tt^{-1}s$.

Proof.

(i) Let $x = st^{-1}t$, then $x \leq s$ and $x \leq t$ since st^{-1} is idempotent. Now let $y \leq s, t$, then $y \leq s$ and $y \leq t$, so we have $y^{-1}y \leq t^{-1}t$ and this implies that $y \leq st^{-1}t = x$. Hence $s \wedge t = x = st^{-1}t$ and one gets $st^{-1} = (st^{-1})^{-1} = ts^{-1}$. Hence, $st^{-1}t = ts^{-1}t$. Through the symmetry also, we have $s \wedge t = ts^{-1}s$ and $s \wedge t = st^{-1}s$.

(ii) Let $a = ss^{-1}t$, then $a \leq s$ and $a \leq t$ since $s^{-1}t$ is idempotent. Now let $b \leq s, t$, then $b \leq s$ and $b \leq t$. Then $bb^{-1} \leq tt^{-1}$ and so $b \leq ss^{-1}t = a$. Hence $s \wedge t = a = ss^{-1}t$ and we have $s^{-1}t = (s^{-1}t)^{-1} = st^{-1}$. Thus, $ss^{-1}t = tt^{-1}s$. By the symmetry we also have $s \wedge t = st^{-1}s$ and $s \wedge t = ts^{-1}t$.

(iii) It follows by (i) and (ii).

4.7.3. Proposition

Suppose k be one either of the three cases $\sim_{\iota}$, $\sim_r$, and $\sim$. Then the following two properties are preserved:

(i) $s\ k\ t$ and $x\ k\ y$ imply that $sx\ k\ ty$.

(ii) $s \leq t$, $x \leq y$ and $t\ k\ y$ imply that $s\ k\ x$.

Proof.

We shall prove the results for $k = \sim_{\iota}$.

(i) Let $s \sim_l t$ and $x \sim_l y$. Then $st^{-1}, xy^{-1} \in E(S)$. Since xy^{-1} is idempotent, $sx(ty)^{-1} = s(xy^{-1})\, t^{-1} \leq st^{-1}$. Hence $sx(ty)^{-1}$ is idempotent and so $s\,x \sim_l ty$. Let $s \sim_r t$ and $x \sim_r y$, then $s^{-1}t, x^{-1}y \in E(S)$. But $(sx)^{-1}\, ty = s^{-1}(x^{-1}y)\, t \leq s^{-1}t$ since $x^{-1}y$ is idempotent. Hence $(sx)^{-1}ty$ is an idempotent, and so $sx \sim_r ty$. By the two previous relations, we find $sx \sim ty$.

(ii) Let $s \leq t$ and $x \leq y$ and $t \sim_l y$, then $sx^{-1} \leq ty^{-1} \in E(S)$. Thus, $s \sim_l y$. Let $s \leq t$ and $x \leq y$ and $t \sim_r y$ and so $s^{-1}x \leq t^{-1}y \in E(S)$. Hence $s \sim_r y$. By the two previous relations, we have $s \sim y$.

4.7.4. Definition

Let S an inverse semigroup and X subset of S, we say X is compatible if any pair of elements in X are compatible.

4.7.5. Proposition

Let S be an inverse semigroup and $s, t \in S$. Then

(i) If $s \sim_l t$ and $s^{-1}s \leq t^{-1}t$ then $s \leq t$.

(ii) If $s \sim_r t$ and $ss^{-1} \leq tt^{-1}$then $s \leq t$.

(iii) [s] is a compatible subset of S.

Proof.

(i) Since $s^{-1}s \leq t^{-1}t$, $s \leq st^{-1}t$. Where st^{-1} is an idempotent and thus $(st^{-1})\, t \leq t$. Then $s \leq t$.

(ii) We have $ss^{-1} \leq tt^{-1}$ this means that $s \leq tt^{-1}s$ and so $s \leq st^{-1}t$. Where st^{-1} is idempotent and thus $(st^{-1})\, t \leq t$. Then $s \leq t$.

(iii) We say [s] is compatible if any pair of elements in [s] are compatible. Suppose $s \leq t$ and $u \leq v$ and $s \sim_l t$. And therefore, $su^{-1} \leq tv^{-1} \in E(S)$. This means that [s] is compatible.

4.7.6. Definition

Let S an inverse semigroup. If $s^{-1}t = 0 = st^{-1}$, we say s, t $\in$S are orthogonal and denoted by s $\perp$ t.

4.8. MEETS AND JOINS

4.8.1. Proposition

Let S be an inverse semigroup and X a non-empty set of idempotents. Then

(i) If $\wedge$ X found, this means idempotent.
(ii) If $\vee$ X found, this means idempotent.

Proof.

(i) Since idempotents form an order ideal, prove holds.
(ii) Let x = $\vee$ X, then e $\leq$ x for each e$\in$X. Thus e $\leq x^{-1}x$ for each e$\in$X.

Hence x $\leq x^{-1}x$, so that x is idempotent.

4.8.2. Note

Let S be an inverse semigroup, for any non-empty subset of S it is possible have a meet, otherwise of joins.

4.8.3. Proposition

Let S be an inverse semigroup and let X be a non-empty subset of S such that $\vee$ X exists. So, any two elements of X are compatible.

Proof.

Let $x, y \in X$. By the definition, we have $x,y \leq \vee X$. Thus $x \sim y$ by Proposition 3.7.4.

4.8.4. Definition

An inverse semigroup is complete if every non-empty compatible subset has a join.

4.8.5. Proposition

Let S be an inverse semigroup and let $X = \{x_i : i \in I\}$ be any non-empty subset of S.

Then

(i) If $\vee x_i$ exists, then $\vee x_i^{-1}x_i$ exists and $(\vee x_i)^{-1} (\vee x_i) = \vee x_i^{-1}x_i$
(ii) If $\vee x_i$ exists, then $\vee x_i x_i^{-1}$ exists and $(\vee x_i) (\vee x_i)^{-1} = \vee x_i x_i^{-1}$

Proof.

(i) Let $x = \vee x_i$. Then $x_i \leq x$ implies $x_i^{-1}x_i \leq x^{-1}x$. Thus, the set $\{x_i^{-1}x_i : i \in I\}$ is bounded above by $x^{-1}x$. Now let that $x_i^{-1}x_i \leq y$ for some $y \in S$ and for each $i \in I$. So, $x_i \leq x_i y \leq xy$ for all $i \in I$. Thus $x = \vee x_i \leq xy$. But then $x = (xx^{-1}) xy = xy$, so that $x^{-1}x = x^{-1}xy$. Hence $x^{-1}x \leq y$. It follows that $\vee x_i^{-1}x_i = x^{-1}x$.
(ii) Let $x = \vee x_i$, then $x_i \leq x$ implies $x_i x_i^{-1} \leq x x^{-1}$. Thus, the set $\{x_i x_i^{-1} : i \in I\}$ is bounded above by $x x^{-1}$. Now let that $x_i x_i^{-1} \leq y$ for some $y \in S$ and for each $i \in I$. So, $x_i \leq x_i y \leq xy$ for all $i \in I$. Thus $x = \vee x_i \leq xy$. But then $x = (x^{-1}x)xy = xy$, so that $x x^{-1} = x x^{-1}y$. Hence $x x^{-1} \leq y$. It follows that $\vee x_i x_i^{-1} = x x^{-1}$.

4.8.6. Proposition

Let S be an inverse semigroup and let $X = \{x_i : i \in I\}$ a non-empty subset of S and $s \in S$.

Then

(i) If $x = \vee x_i$ and $x_i x_i^{-1} \leq s^{-1}s$ for each $i \in I$ then $\vee sx_i$ exists and $sx = \vee sx_i$.

(ii) If $x = \vee x_i$ and $x_i^{-1}x_i \leq ss^{-1}$ for each $i \in I$ then $\vee x_is$ exists and $xs = \vee x_is$.

Proof.

(i) Since $x_i \leq x$ for each $i \in I$, we have $sx_i \leq sx$ for each $i \in I$. Thus the set $\{sx_i : i \in I\}$ is bounded above by sx. Now let that $sx_i \leq y$ for some $y \in S$ and for each $i \in I$. Then $s^{-1}sx_i \leq s^{-1}y$ and so $x_i \leq s^{-1}y$ since $x_ix_i^{-1} \leq s^{-1}s$. Thus $x \leq s^{-1} y$ and so $sx \leq ss^{-1}y \leq y$. It follows that $\vee sx_i = sx$.

(ii) Since $x_i \leq x$ for each $i \in I$, we have $x_is \leq xs$ for each $i \in I$. Thus, the set $\{x_is : i \in I\}$ is bounded above by xs. Now let that $x_is^{-1} \leq y$ for some $y \in S$ and for each $i \in I$. Then $sx_is^{-1} \leq sy$ and so $x_i \leq sy$ since $x_i^{-1}x_i \leq ss^{-1}$. Thus $x \leq sy$ and so $s^{-1}x \leq ss^{-1}y \leq s^{-1} y$. It follows that $\vee x_is = xs$.

4.8.7. Proposition

Let S be an inverse semigroup and let $X = \{x_i : i \in I\}$ a non-empty subset of S and $s \in S$.

Then

(i) If $x = \wedge x_i$ exists, then $\wedge sx_i$ exists and $\wedge sx_i = sx$.

(ii) If $x = \wedge x_i$ exists, then $\wedge x_is$ exists and $\wedge x_is = xs$.

Proof.

(i) By the definition, $x \leq x_i$ for each $i \in I$, and thus $sx \leq sx_i$ for each $i \in I$. So the set $\{sx_i : i \in I\}$, it is bounded from below by sx. Also, suppose that $y \leq sx_i$ for some $y \in S$ and for each $i \in I$. Then $s^{-1}y \leq s^{-1} sx_i \leq x_i$, thus that $s^{-1}y \leq x$.

Hence $ss^{-1}y \leq sx$. Now $y \leq sx_i$ and so $yy^{-1} \leq (sx_i)(sx_i)^{-1} = sx_ix_i^{-1}s^{-1} \leq ss^{-1}$. Thus, $ss^{-1}y = y$, and so $y \leq sx$. It follows that $\wedge\, sx_i = sx$.

(ii) By the definition, $x \leq x_i$ for each $i \in I$, and thus $xs \leq x_is$ for each $i \in I$. So, the set $\{x_is\colon i \in I\}$, it is bounded from below by xs. Also, suppose that $y \leq x_is$ for some $y \in S$ and for each $i \in I$. Then $ys^{-1} \leq x_iss^{-1} \leq x_i$, thus that $ys^{-1} \leq x$. Hence $ys^{-1}s \leq xs$. Now $y \leq x_is$ and so $y^{-1}y \leq (sx_i)^{-1}(sx_i) = s^{-1}x_i^{-1}x_is \leq s^{-1}s$. Thus, $s^{-1}sy = y$, and so $y \leq xs$. It follows that $\wedge\, x_is = xs$.

4.8.8. Definition

Let S be an inverse semigroup and A is a non-empty subset of S. If A has $\vee A$, then $\vee sA$ is present $s(\vee A) = \vee sA$ for every element $s \in S$, then S is called be left infinitely distributive. The infinitely distributed left and right a semigroup are called infinitely distributed.

4.9. HOMOMORPHISMS

Homomorphisms between inverse semigroups are just semigroup homomorphisms. If $(A, \leq)$ and $(A', \leq)$ are possets, then a function $\theta\colon A \rightarrow A'$ is called order-preserving if $x \leq y$ so that $\theta(x) \leq \theta(y)$.

4.9.1. Proposition

Let $\theta\colon S \rightarrow T$ be a homomorphism between two inverse semigroups S and T.

Then

(i) $\theta(s^{-1}) = \theta(s)^{-1}$ for all $s \in S$.

(ii) If e is an idempotent element, then $\theta(e)$ is idempotent.

(iii) If $\theta(s)$ is an idempotent element, then there exists an idempotent e in S such that $\theta(s) = \theta(e)$.

(iv) $\mathrm{Im}\theta$ is an inverse subsemigroup of T.

(v) If U is an inverse subsemigroup of T, then $\theta^{-1}(U)$ is an inverse subsemigroup of S.

(vi) The function θ is order- preserving.

(vii) Let $x, y \in S$ such that $\theta(x) \leq \theta(y)$. Then there exists an element $x' \in S$ such that $x' \leq y$ and $\theta(x') = \theta(x)$.

Proof.

(i) We have $\theta(s)\theta(s^{-1})\theta(s) = \theta(s)$ and $\theta(s^{-1})\theta(s)\theta(s^{-1}) = \theta(s^{-1})$. Thus by uniqueness of inverses we have that $\theta(s^{-1}) = \theta(s)^{-1}$.

(ii) $\theta(e)^2 = \theta(e)\theta(e) = \theta(e)$.

(iii) If $\theta(s)^2 = \theta(s)$, then $\theta(s^{-1}s) = \theta(s^{-1})\theta(s) = \theta(s)^{-1}\theta(s) = \theta(s)^2 = \theta(s)$.

(iv) Since θ is a semigroup homomorphism, $\mathrm{im}\theta$ is a subsemigroup of T. By (i), $\mathrm{im}\theta$ is closed under inverses.

(v) It is clear, immediately.

(vi) Suppose that $x \leq y$. Then $x = ye$ for some idempotent e. Then $\theta(x) = \theta(y)\theta(e)$ and $\theta(e)$ is idempotent. Hence $\theta(x) \leq \theta(y)$.

(vii) let $x' = yx^{-1}x$. Then $x' \leq y$, and $\theta(x') = \theta(y)\theta(x^{-1}x) = \theta(x)$.

Let $\theta: S \to A$ be covering homomorphism; we call that S is a covering of A.

Let $\theta: B \to A$ be a perfect homomorphism, and B is an inverse subset of S, we call that A divides S. For there to be a monoid homomorphism, a homomorphism between monoids is necessary to preserve identities. Same a way, if zero semigroups are to have homomorphism, a homomorphism between zero semigroups is required to preserve the zeros. The similarity from an inverse semigroup S to a symmetric inverse monoid is called the representation of S by partial bisection. And if A homomorphism is injective, then the representation is called safe.

The homomorphism $\phi: S \to T$ between inverse semigroups means that for each subset $A \subseteq S$ in which $\vee A$ exists, $\vee \phi(A)$ in T and $\phi(\vee A) = \vee(\phi(A))$ cases it is said to preserve the concatenation. For each subset $A \subseteq S$ such as $\wedge A$ exists, homomorphism is said to be conserving if T has $\wedge \phi(A)$ and $\phi(\wedge A) = \wedge(\phi(A))$.

PART 5

THE RELATIONSHIP BETWEEN INVERSEMIGROUPS AND THE LEAVITT PATH ALGEBRA

In this proposal we study formal presentation of inverse semigroups built from directed graphs. Thus, this study is referred as Leavitt inverse semigroups. In this chapter, we explain the structure of the Leavitt inverse semigroup, these semigroups are strongly related with the graph inverse semigroups and Leavitt path algebras. We also introduce a class for the Leavitt inverse semigroup of a graph. We refer to LI(E) as a multiplicative subsemigroup of $L_F(E)$ produced by $E^0 \cup E^1 \cup (E^1)^*$. The main reference for this chapter is [8].

5.1. THEOREM

If p is a directed path in E and $v \in E^0$, then the following elements are basis for Leavitt path algebra L_F (E):

(i) v, p, p^*,

(ii) pq^* where $p = e_1 ... e_n$, $q = f_1 ... f_m$, $e_i, f_j \in E^1$, $r(e_n) = r(f_m)$, and either $e_n \neq f_m$ or $e_n = f_m$ but this edge $e_n = f_m$ is not special.

5.2. THEOREM

Let E graph, LI(E)is an inverse semi-group. If LI(E) $\cong$L(E), then.

(i) pq^* where $p = e_1 ... e_n$, $q = f_1 ... f_m$ are (maybe empty) directed paths with $r(e_n) = r(f_m)$ and $e_n \neq f_m$.

(ii) $pq^* = p'ee^*q'^*$ where $p'$and q' are (maybe empty) directed paths with $r(p') = r(q')$ and the vertex $s(e) = r(p') = r(q')$ has out-degree ≥ 2.

5.3. DEFINITION

Let $p = e_1e_2...e_n$ be a directed path in a graph E. If at least one of the $s(e_i)$ has out-degree greater than 1, then p exits. Particularly, an edge $e \in E^1$exits if and only if $s(e)$ has out-degree greater than 1. Also, the directed path $p = e_1e_2...e_n$ has no exits (NE) if every vertex $s(e_i)$, i =1, ..., n has out-degree 1. Moreover, the empty path is defined at any vertex ν to be an NE path.

5.4. THEOREM

For any graph E, the elements of the non-zero idempotents of LI(E) are defined by pp^*. Moreover $pp^* = qq^*$ in LI(E) if and only if either $q = pp_1$ for NE path p_1 or $p = qq_1$ for NE path q_1. Especially, $pp^* = \nu$ in LI(E)for some $\nu \in E^0$ if and only if $\nu = s(p)$ and p is an NE path.

5.5. THEOREM

Let E graph, the congruence $\leftrightarrow$is the nucleus of the natural homomorphism that is LI(E) $\cong$ I(E)/$\leftrightarrow$.

Proof.

We have that the nucleus of the natural homomorphism from I(E) to LI(E)is the congruence ρ defined a relation by $\{(ee^*, s(e)) : s(e)$has out-degree 1in $E^0\}$.Let e be an edge of E, $s(e)$ has out-degree1.Then s(e) ρ ee^*. Let $0<x\leq s(e)$ in I(E). Then $x = pp^*$ for some directed path p with $s(p) = s(e)$. As for $p = s(e)$or $p = eq$ for directed path q with $s(q) = r(e)$. It is clear that $s(e)^{\downarrow} \cap (ee^*)^{\downarrow} \neq \{0\}$.Also, if $p = eq$, we have $(pp^*)^{\downarrow} = \{eqtt^*q^*e^* : s(t) = r(q)\}$,so $(pp^*)^{\downarrow} \cap (ee^*)^{\downarrow} = \{0\}$.Thus $s(e) \rightarrow ee^*$. Also, since $ee^* \leq s(e)$, $ee^* \rightarrow s(e)$. Thus, $s(e) \leftrightarrow ee^*$ and so $\rho \subseteq \leftrightarrow$. $p_1q_1^*$, $p_2q_2^*$, are non-zero elements of I(E) so $p_1q_1^* \leftrightarrow p_2q_2^*$. Then $(p_1q_1^*)^{\downarrow} \cap (p_2q_2^*)^{\downarrow} \neq \{0\}$, so there exists paths t_1,t_2 such that $p_1t_1t_1^*q_1^* = p_2t_2t_2^*q_2^*$. As for p_1 is a pseudo of p_2 or p_2 is a pseudo of p_1. Let d is an NE path, $p_2 = p_1d$ for some path $d = e_1e...e_n$. If not, then $1 \leq i \leq n$ such that $s(e_i)$has out-degree ≥ 2, and so there is some edge f with $f \neq e_i$ and $s(f) = s(e_i)$. Let $d_1 = e_1...e_{i-1}$and

$d_2=e_1...e_n$. Then $(p_1d_1ff^*d_1q_1^*)^{\downarrow}\cap(p_2q_2^*)^{\downarrow}\neq\{0\}$.Hence there exist pathst$_3$,t$_4$suchthat $p_1d_1f\mathrm{t}_3\mathrm{t}^*{}_3f^*d_1q_1^*=p_2\mathrm{t}_4\mathrm{t}_4^*q_2^*= p_1d_1d_2\mathrm{t}_4\mathrm{t}_4^*q_2^*$.This implies that $d_2\mathrm{t}_4=f\ \mathrm{t}_3$. However, this is not possible because the first edge of d_2 is $e_i\neq f$. Therefore, d is an NE path. One gets $p_1\mathrm{t}_1\mathrm{t}_4^*q_1^*= p_2\mathrm{t}_2\mathrm{t}_2^*q_2^*= p_1d\mathrm{t}_2\mathrm{t}_2^*q_2^*$, so $\mathrm{t}_1=d\mathrm{t}_2$ and hence $p_2\mathrm{t}_2= p_1\mathrm{t}_1= q_1d\mathrm{t}_2$.This means that $q_2= q_1\mathrm{d}$. So $p_2= p_1d$ and $q_2= q_1d$ for some NE path d. Thus, $(p_1q_1^*)\ \rho\ (p_2q_2^*)$. Hence $\leftrightarrow\subseteq\rho$. So, the $\leftrightarrow$ and ρ coincide.

5.6. NOTE

A graph E receives a directed immersion into a circle R1if and only if all the vertices have out-degree ≤ 1.

5.7. DEFINITION

Let X be a set and G a group, then the Brandt semigroup, denoted $B_X(G)$, is a semigroup and defined as

$B_X(G) = \{(x_1, g, y_1): x_1, y_1 \in X, g \in G\} \cup \{0\}$
with multiplication $(x_1, g, y_1)\ (x_2, h, y_2) = (x_1, gh, y_2)$ if $y_1 = x_2$ and 0 otherwise.

5.8. THEOREM

Let E be a connected graph which immerse into a circle, then the followings are hold:

(i) If E is a tree, then $LI(E) \cong B_{E^0}(1)$, the combinatrorial $|E^0|\times|E^0|$ is a Brandt semigroup.
(ii) If E is not a tree, then $LI(E) \cong B_{E^0}(\mathbb{Z})$, the $|E^0|\times|E^0|$ is a Brandt semigroup with maximal subgroups isomorphic to $\mathbb{Z}$.

5.9. THEOREM

Let E and Δ be two connected graphs which immerse into a circle and F a field. Then the followings are equivalent.

(i) LI(E) is isomorphic to LI(Δ);

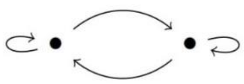
(ii) $L_F(E)$ is isomorphic to $L_F(\Delta)$;

(iii) $|E^0| = |\Delta^0|$ and either E and Δ are both trees or $\pi_1(E) \cong \pi_1(\Delta) \cong Z$.

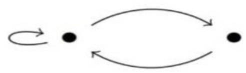
In the following example, it is shown that if $L_F(E)$ is isomorphic to $L_F(\Delta)$, then LI(E) may not be isomorphic to LI(Δ):

5.10. EXAMPLE

Let E_1 and E_2 be two graphs as follows:



E_1:



E_2:

It is easy to see that $L_F(E_1) \cong L_F(E_2)$, and we see from [2, Theorem 6.12] the corresponding Leavitt inverse semigroups are not isomorphic.

5.11. THEOREM

Let E any graph, then the followings hold:

(i) E^0 is the set of maximal idempotents in LI(E).

(ii) $\{pee^* p^* : p$ is an NE path, $e \in E^1$ and the out degree of $s(e) \geq 2\}$ is the set of maximal idempotents of LI(E)$\setminus E^0$.

5.12. THEOREM

Let E and Δ be two graphs and θ an isomorphism between the Leavitt inverse semigroups LI(E) and LI(Δ), then we have the followings:

(i) $\theta\ (\nu) \in \triangle^0$ for all $\nu \in E^0$

(ii) for each nonzero $pq^* \in$ LI(E), if $\theta\ (pq^*) = p_1q_1^*$ and q is an NE path, then q_1 is an NE path, $\theta\ (s(p)) = s(p_1)$, $\theta\ (pp^*) = p_1\, p_1{}^*$ and $\theta\ (s(q)) = q_1q_1^* = s(q_1)$.

(iii) for every nonzero $pq^* \in$LI(E), if $\theta\ (pq^*) = p_1q_1^*$ and p, q are NE paths, then p_1, $q_1$are NE paths, $\theta\ (s(p)) = p_1\, p_1{}^* = s(p_1)$ and $\theta\ (s(q)) = q_1q_1^* = s(q_1)$.

(iv) for any $e \in E^1$, if $s(e)$ has out-degree ≥ 2, then there is NE paths p_1, p_2, p_3 and an edge $\check{e}$ for which $s(\check{e})$ has out-degree ≥ 2 thus $\theta\ (e) = p_1\check{e}p_2p_3^*$ and there is NE paths q_1, q_2 , q_3 such that $\theta^{-1}(\check{e}) = q_1eq_2\, q_3^*$.

(v) for any $\nu \in E^0$, if $s^{-1}(\nu) = \{e_1, \ldots, e_n\}$ with $n \geq 2$, there is NE paths p, p_i, q_i and distinct edges $\check{e}_i$, $i = 1,\ldots., n$ thus $\theta\ (e_i) = p\check{e}_ip_iq_i^*$, $i = 1,\ldots., n$ and $s^{-1}(r(p)) = \{\check{e}_1, \ldots, \check{e}_n\}$.

5.13. THEOREM

Let E and Δ be two connected graphs and F a field. If LI(E) is isomorphic to LI(Δ), then $L_F(E)$) is isomorphic to $L_F(\Delta)$.

Proof.

We have $L_F(E)$is isomorphic to the quotient of the contracted semigroup algebra F_0LI(E)of LI(E) by the ideal I_1 defined by elements $\sum_{e\in s-1(\nu)} ee* - \nu$ for $\nu \in E^0$ with the out-degree of $\nu \geq 2$. L_F (Δ)is isomorphic to the contracted semigroup algebra F_0LI(Δ)of LI(Δ)by the ideal I_2 defined by elements of the form $\sum_{d\in s-1(u)} dd* - u$ for $u \in E^0$with the out-degree of $u \geq 2$.

Let θ be an isomorphism from LI(Γ) to LI(Δ). So, θ is an algebra isomorphism, say η, from F_0LI(E) to F_0LI(Δ). For $\nu \in E^0$ with the out-degree $\geq$ 1and any $e_i \in s^{-1}(\nu)$, by

Theorem 4.12., there is an NE paths p, p_i, q_i and edges $\check{e}_i \in s^{-1}(r(p))$ such that $\theta(e_i) = p\check{e}_i p_i q_i^*$, $\theta(v) = s(p)$ and $|s^{-1}(v)| = |s^{-1}(r(p))|$. Distinct e_i correspond to distinct $\check{e}_i$. So,

$$\begin{aligned} \eta\left(\sum_{e \in s-1(v)} ee* - v\right) &= \sum_{e \in s-1(v)} \theta(e_i)(\theta(e_i))^* - s(p) \\ &= \sum_{\check{e}i \in s-1(u)} p\check{e}_i\check{e}_i^* p^* - pp^* \\ &= p\left(\sum_{\check{e}i \in s-1(u)} \check{e}_i\check{e}_i^* - u\right) p^* \in I_2 \end{aligned}$$

Hence $\eta(I_1) \subseteq I_2$.Likewise $\eta^{-1}(I_2) \subseteq I_1$. So, one gets $\eta(I_1) = I_2$ and $\eta^{-1}(I_2) = I_1$. Hence $L_F(E)$) is isomorphic to $L_F(\Delta)$.

PART 6

CONCLUSION

In this thesis, we studied a class of inverse semigroups built from the Leavitt path algebras. In the beginning of the study, we gave our consideration to the Leavitt path algebras. To understand this nature, we firstly discussed on directed graphs and its properties. In the following, we examined the role of inverse semigroups in algebra. Thus, we investigated its structures, ideals and homomorphisms in details. In the last chapter, we analyzed the class of inverse semigroups related to the Leavitt path algebras. We studied a presentation for the Leavitt inverse semigroups and defined the structure of the Leavitt inverse semigroups.

REFERENCES

[1] Abrams, G., Ara P., and Molina M. S., “Leavitt path algebras, ser. lecture notes in math.”, London, England: Springer, vol. 2191, (2017).

[2] Lawson, M. V., “Inverse semigroups the theory of partial symmetries”, University College of North Wales, World Scientific, (1998).

[3] Wagner, V. V., Generalized groups, Doklady Akademii Nauk SSSR, 84, 1119-1122, (1952).

[4] Preston, G. B., “Inverse semigroups”, Journal of the London Mathematical Society, 29, 396-403, (1954).

[5] Clifford, A. H., “Semigroups admitting relative inverses”, Annals of Mathematics, 42,1037-1049, (1941).

[6] Rees, D., “On the groups of a set of partial transformation”, Journal of the London Mathematical Society, 22, 281-284, (1947).

[7] Clifford, A. H., “A class of d-simple semigroups”, American Journal of Mathematics, 75, 547-556, (1953).

[8] Meakin, J., Milan, D., Wang, Z., “On class of inverse semigroups related to Leavitt path algebras”, Advances in Mathematics, 384, 107729, (2021).

[9] Preston, G. B., “Inverse semigroups with maximal right ideals”, Journal of the London Mathematical Society, 29, 404-411, (1954).

[10] Preston, G. B., “Represantation of inverse semigroups”, Journal of the London Mathematical Society, 29, 411-419, (1954).

[11] Rees, D., “On the ideal structure of a semigroup satisfying a cancellation law”, Quaterly Journal of Mathematics (Oxford), 19, 101-108, (1948).

[12] Clifford, A. H., “Matrix representation of completely simple semigroups”, American Journal of Mathematics, 64, 327-342, (1942).

[13] Moreno, Ara, P., and Pardo, M. A., E., ‘‘Nonstable K-theory for graph algebras, Algebras and Representation Theory”, 10, 157-178, (2007).

[14] Abrams, G., and Pino, A. G., ‘‘The Leavitt path algebras of a arbitrary graphs”, Hauston J., Math, 34(2), 423-442, (2008).

[15] Abrams, G., and Aranda Pino, G., ‘‘The Leavitt path algebra of a graph”, J., Algebra, 293(2), 319-334, (2005).

[16] Abrams, G., Aranda Pino, G., ‘‘Purely Infinite Simple Leavitt Path Algebras”, Pure Appl, J., Algebra, 207(3), 553–563, (2006).

[17] Aranda, G., Martin Barquero, D., Martin Gonzalez, C., Siles Molina, M., ‘‘The Socle of a Leavitt Path Algebra”, Pure Appl, J., Algebra, 212, 500–509, (2008).

[18] Abrams, G., Aranda Pino, G., Sils Molina, M., ‘‘Finite Dimensional Leavitt Path Algebras”, Pure Appl, J., Algebra, 209, 753–762, (2007).

[19] Tomforde, M., ‘‘Leavitt Path Algebras with Coefficients in a Commutative Ring”, Pure Appl, J., Algebra, 215, 471–484, (2011).

[20] Larki, H., ‘‘Ideal Structure of Leavitt Path Algebras with Coefficients in a Unital Commutative Ring”, Communications in Algebra, 43, 5031–5058, (2015).

RESUME

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