MONOTONICITY RESULTS FOR DISCRETE CAPUTO-FABRIZIO FRACTIONAL OPERATORS

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I certify that, in my opinion, the thesis submitted by Waad Shaban MAHW titled "MONOTONICITY RESULTS FOR DISCRETE CAPUTO-FABRIZIO FRACTIONAL OPERATORS" is fully adequate in scope and quality as a thesis for the degree of Master of Science.

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Waad Shaban MAHW

ABSTRACT

Master Thesis

MONOTONICITY RESULTS FOR DISCRETE CAPUTO-FABRIZIO FRACTIONAL OPERATORS

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The Department of Mathematics

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Nearly every theory in mathematics has a discrete equivalent that simplifies it theoretically and practically so that it may be used in modeling real-world issues. With discrete calculus, for instance, it is possible to find the "difference" of any function from the first order up to the n-th order. On the other hand, it is also feasible to expand this theory using discrete fractional calculus and make n any real number such that the 1⁄2-order difference is properly defined. This thesis is divided into five chapters, each of which develops the most straightforward discrete fractional variational theory while illustrating some fundamental concepts and features of discrete fractional calculus. It is also investigated how the idea may be applied to the development of tumors.

The first chapter provides a succinct introduction to discrete fractional calculus and several key mathematical concepts that are utilized often in the subject. We

demonstrate in Chapter 2 that if the Caputo-Fabrizio nabla fractional difference operator $({}_{a-1}^{CFR}\nabla^{\alpha}y)(t)$ of order $0 < \alpha \leq 1$ and commencing at $a - 1$ is positive for $t = a, a + 1, \ldots,$ then $y(t)$ is α-increasing.

On the other hand, if $y(t)$ is rising and $y(a) \geq 0$, then $({}_{a-1}^{CFR}\nabla^{\alpha}y)(t) \geq 0$.

Additionally, a result of monotonicity for the Caputo-type fractional difference operator is established. We show a fractional difference version of the mean-value theorem as an application and contrast it to the traditional discrete fractional instance.

Keywords : Discrete fractional calculus, discrete exponential kernel, Caputo fractional difference, Riemann fractional difference, discrete fractional mean value theorem.

Science Code : 20406

ÖZET

Yüksek Lisans Tezi

AYRIK CAPUTO-FABRIZIO KESİRLİ OPERATÖRLER İÇİN MONOTONLUK SONUÇLARI

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Matematikteki neredeyse her teorem, teorik ve pratik olarak basitleştiren ayrık bir eşdeğere sahiptir, böylece gerçek dünya sorunlarının modellenmesinde kullanılabilir. Örneğin, ayrık hesapla (Kalkulus), herhangi bir fonksiyonun birinci mertebeden n'inci mertebeye kadar olan "farkını" bulmak mümkündür.

Diğer yandan, ayrık kesirli hesap kullanarak bu teoriyi genişletmek ve 1/2 mertebeden fark uygun şekilde tanımlanacak şekilde herhangi bir gerçel sayı ya da reel sayı yapmak da mümkündür.

Bu tez beş bölüme ayrılmıştır, her bölüm ayrık kesirli hesabın bazı temel kavramlarını ve özelliklerini gösterirken en basit ayrık kesirli varyasyon teorisini geliştirir. Ayrıca, fikrin tümörlerin gelişimine nasıl uygulanabileceği de araştırılmıştır.

İlk bölüm ayrık kesirli hesabı ve bu konuda sıklıkla kullanılan birkaç temel matematiksel kavramı tanıtmaktadır. Bölüm 2'de, 0<α≤1 mertebesindeki ve a-1'de başlayan Caputo-Fabrizio nabla kesirli fark operatörü (〖_(a-1)^CFR〗 ∇^α y)(t), t = a, a + 1, ... için pozitifse, o zaman y(t) α -artar.

Diğer yandan, y(t) yükseliyorsa ve y(a)≥0 ise, (〖_(a-1)^CFR〗 ∇^α y)(t)≥0.

Ayrıca, Caputo tipi kesirli fark operatörü için monotonluğun bir sonucu elde edilmiştir. Bir uygulama olarak ortalama değer teoreminin kesirli bir fark versiyonu gösterilmiştir ve onu geleneksel ayrık kesirli örnekle karşılaştırılmıştır.

Anahtar Sözcükler : Ayrık kesirli hesap, ayrık üstel çekirdek, Caputo kesirli fark, Riemann kesirli fark, ayrık kesirli ortalama değer teoremi.

Bilim Kodu : 20406

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LIST OF ABBREVIATIONS

$\Gamma(\alpha)$: Gamma function.

∇ : Backward (nabla) difference operator.

∇_a^α : Backward (nabla) difference operator of order α with start point a.

${}^{CFC}\nabla_a^\alpha$: Caputo-Fabrizio difference in the Caputo sense

${}^{CFR}\nabla_a^\alpha$: Caputo-Fabrizio difference in the Riemann sense

${}^{CFR}\nabla_a^{-\alpha}$: Fractional sum

PART 1

INTRODUCTION

In numerous fields of engineering and research over the past ten years, the fractional calculus has been successfully applied [1],[2]. Discrete fractional calculus (DFC) was successfully developed using the fundamental ideas of this type of nonlocal calculus [3],[4]. This new direction, which was started more than 10 years ago, is in a state of steady evolution, and it has just recently started to be recognized as a potent instrument for uncovering hitherto unknown dynamics of intricate discrete dynamical systems. The discrete diffusion equation included in the discrete Riesz derivative was one of the most recent discoveries. The discrete diffusion equation included inside the discrete Riesz derivative [5], [6]. Therefore, the discrete fractional calculus may be a natural development of the conventional discrete ones. And Fabrizio, Caputo[7] On the basis of a nonsingular kernel, a different fractional derivative was presented. This operator's discrete variant was described in [8]. We think that the appearance of various forms of memory kernels improves the likelihood that various kinds of models will be appropriately developed when various types of memory emerge. Recent research has looked into discrete functions' discrete fractional operators to examine the monotonicity properties of such functions. While others looked into fractional difference operators of order $\alpha > 1$ [9],[10], some writers addressed the monotonicity analysis of fractional difference operators of orders $0 < \alpha < 1$, such as delta- or nabla-types [11]. These novel findings motivate us to compare the monotonicity results for this discrete fractional operator with discrete exponential kernel to the discrete classical ones and discuss them in this thesis. We think that the fractional differences considered in this thesis result in novel kernels with new memories, which might be of diverse importance for applications. These kernels differ from the standard nabla fractional differences with kernels relying on the increasing factorial powers.

1.1. PRELIMINARIES

Discrete fractional calculus's fundamental concepts and results are provided in the next chapter. The fractional sum and the fractional difference of a function $f(x)$ to a random order α, starting from a, will be denoted by $\nabla_a^{-\alpha} f(x)$ and $\nabla_a^{-\alpha} f(x)$ respectively. Where α is a real number that is positive, and for a real number a, we demoted $\mathbb{N}_a = \{a, a+1, a+2, ...\}$. Our recommendation for our readers is the reference [28] for further information on discrete fractional calculus concepts.

The nabla discrete exponential kernel may be expressed using the time scale notation as $(1-\alpha)^{t-\rho(s)}$ where $\rho(s) = s - 1$ [29].

1.1.1. Caputo Fractional Difference

The Caputo-Fabrizio in the Caputo sense nabla difference of f may be defined as follows for $0 < \alpha < 1$ and f defined on $\mathbb{N}_a$:

$$({}^{CFC}\nabla_a^{\alpha} f)(t) = \frac{B(\alpha)}{1-\alpha} \sum_{s=a+1}^{t} (\nabla_s f)(s)(1-\alpha)^{t-\rho(s)}$$

$$= B(\alpha) \sum_{s=a+1}^{t} (\nabla_s f)(s)(1-\alpha)^{t-s} \quad (1.1)$$

where $B(\alpha)$ is a normalizing positive constant which depends on α and sustaining $B(0) = B(1) = 1$[8].

1.1.2. Riemann Fractional Difference

For $0 < \alpha < 1$ and f defined on $\mathbb{N}_a$, the Caputo-Fabrizio in the Riemann sense nabla difference of f can be defined by:

$$\left({}^{CFR}\nabla_a^{\alpha}f\right)(t) = \frac{B(\alpha)}{1-\alpha}\nabla_t \sum_{s=a+1}^{t} f(s)(1-\alpha)^{t-\rho(s)}$$
$$= B(\alpha)\nabla_t \sum_{s=a+1}^{t} f(s)(1-\alpha)^{t-s} \qquad (1.2)$$

wherever $B(\alpha)$ is a normalizing positive constant depending on α and satisfying $B(0) = B(1) = 1$[8].

1.1.3. Fractional Sum

For $0 < \alpha < 1$ and f defined on $\mathbb{N}_a$, the fractional sum of f can be defined by:

$$\left({}^{CF}\nabla_a^{-\alpha}f\right)(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)}\sum_{s=a+1}^{t} f(s)ds \qquad (1.3)$$

In [8], it was shown that $\left({}^{CF}\nabla_a^{-\alpha} \quad {}^{CF}\nabla_a^{\alpha}f\right)(t)$. Also, it was shown that $\left({}^{CF}\nabla_a^{\alpha} \quad {}^{CF}\nabla_a^{-\alpha}f\right)(t)$.

The following statement and lemma include several elements that are crucial to moving forward.

1.1.4. Proposition

The association between Riemann and Caputo kind fractional difference given by [12].

$\left({}^{CFC}\nabla_a^{\alpha}f\right)(t) = \left({}^{CFR}\nabla_a^{\alpha}f\right)(t) - \frac{B(\alpha)}{1-\alpha}f(a)(1-\alpha)^{t-a}$.

1.1.5. Lemma

For $\alpha \in (0,1)$ and g defined on $\mathbb{N}_a$, there are

(i) $\left({}^{CF}\nabla_a^{-\alpha}(1-\alpha)^t\right)(t) = \frac{(1-\alpha)^{a+1}}{B(\alpha)}$ (1.4)

(ii) $\nabla_s(1-\alpha)^{t-s} = \alpha(1-\alpha)^{t-s}$

(iii) $\left({}^{CF}\nabla_a^{-\alpha}\nabla g\right)(t) = \left(\nabla^{CF}\nabla_a^{-\alpha} g\right)(t) - \frac{\alpha}{B(\alpha)} g(a)$

(iv) $\nabla(1-\alpha)^t = -\alpha(1-\alpha)^{t-1}$

(v) $\left({}^{CFR}\nabla_a^{\alpha}(1-\alpha)^t\right)(t) = B(\alpha)(1-\alpha)^{t-1}[1-\alpha(t-a)]$

Proof

Strat with the proof of (i)

$$\left({}^{CF}\nabla_a^{-\alpha}(1-\alpha)^t\right)(t) = \frac{1-\alpha}{B(\alpha)}(1-\alpha)^t + \frac{\alpha}{B(\alpha)}\sum_{s=a+1}^{t}(1-\alpha)^s ds$$

Since $\alpha \in (0,1)$ we can apply geometric series, and we get

$$= \frac{1-\alpha}{B(\alpha)}(1-\alpha)^t + \frac{\alpha}{B(\alpha)}(1-\alpha)^{a+1}\frac{1-(1-\alpha)^{t+1-(a+1)}}{1-(1-\alpha)}$$

$$= \frac{1-\alpha}{B(\alpha)}(1-\alpha)^t + \frac{\alpha}{B(\alpha)}(1-\alpha)^{a+1}\frac{1-(1-\alpha)^{t-a}}{\alpha}$$

$$= \frac{1}{B(\alpha)}[(1-\alpha)^{t+1} + (1-\alpha)^{a+1} - (1-\alpha)^{t+1}]$$

$$= \frac{(1-\alpha)^{a+1}}{B(\alpha)} \tag{1.5}$$

The proof of (ii)

$$\nabla_s(1-\alpha)^{t-s} = (1-\alpha)^{t-s} - (1-\alpha)^{t-(s-1)}$$

$$= (1-\alpha)^{t-s} - (1-\alpha)^{t-s+1}$$

$$= (1-\alpha)^{t-s+1}[\frac{1}{(1-\alpha)} - 1]$$

$$= (1-\alpha)^{t-s+1}[\frac{1-1+\alpha}{(1-\alpha)}]$$

$$= \alpha(1-\alpha)^{t-s}.$$

The proof of (iii)

$$({}^{CF}\nabla_a^{-\alpha}\nabla g)(t) = \frac{1-\alpha}{B(\alpha)}\nabla g(t) + \frac{\alpha}{B(\alpha)}\sum_{s=a+1}^{t}\nabla g(s)ds.$$

But note that $\sum_{s=a+1}^{t}\nabla g(s)ds = g(t) - g(a)$, so we can write

$$= \frac{1-\alpha}{B(\alpha)}\nabla g(t) + \frac{\alpha}{B(\alpha)}[g(t) - g(a)].$$

And from $\nabla\sum_{s=a+1}^{t} g(s)ds = g(t)$, we get

$$= [\frac{1-\alpha}{B(\alpha)}\nabla g(t) + \frac{\alpha}{B(\alpha)}\nabla\sum_{s=a+1}^{t} g(s)ds] - \frac{\alpha}{B(\alpha)}g(a)$$
$$= \nabla[\frac{1-\alpha}{B(\alpha)}g(t) + \frac{\alpha}{B(\alpha)}\sum_{s=a+1}^{t} g(s)ds] - \frac{\alpha}{B(\alpha)}g(a)$$
$$= (\nabla^{CF}\nabla_a^{-\alpha}g)(t) - \frac{\alpha}{B(\alpha)}g(a).$$

The prove of (iv)

$\nabla(1-\alpha)^t = (1-\alpha)^t - (1-\alpha)^{t-1}$
$= (1-\alpha)^{t-1}[(1-\alpha) - 1]$
$= (1-\alpha)^{t-1}[(1-\alpha) - 1]$
$= -\alpha(1-\alpha)^{t-1}.$

The prove of (v)

$$({}^{CFR}\nabla_a^{\alpha}(1-\alpha)^t)(t) = B(\alpha)\nabla_t\sum_{s=a+1}^{t}(1-\alpha)^s(1-\alpha)^{t-s}$$
$$= B(\alpha)\nabla_t(1-\alpha)^t\sum_{s=a+1}^{t}1$$

$$
\begin{aligned}
&= B(\alpha)\nabla_t[(1-\alpha)^t(t-a)] \\
&= B(\alpha)[(1-\alpha)^t(t-a) - (1-\alpha)^{t-1}(t-1-a)] \\
&= B(\alpha)(1-\alpha)^{t-1}[(1-\alpha)(t-a) - (t-1-a)] \\
&= B(\alpha)(1-\alpha)^{t-1}[t-a-t\alpha+a\alpha-t+a+1] \\
&= B(\alpha)(1-\alpha)^{t-1}[1-t\alpha+a\alpha] \\
&= B(\alpha)(1-\alpha)^{t-1}[1-\alpha(t-a)]. \qquad (1.6)
\end{aligned}
$$

PART 2

THE MONOTONICITY RESULT

2.1. α-INCREASE

Let y be a function defined on $\mathbb{N}_{\boldsymbol{a}}$ so that satisfying $y(a) \geq 0$. Then y is named an α-increasing function on $\mathbb{N}_{\boldsymbol{a}}$ if [13]

$$y(t+1) \geq \alpha y(t) \quad for\ all\ t \in \mathbb{N}_{\boldsymbol{a}}.$$

2.1.1. Theorem

Let y be a function defined on $\mathbb{N}_{\boldsymbol{a-1}}$, if

$\alpha \in (0,1)$, and

$$({}^{CFR}\nabla_{a-1}^{\alpha} y)(t) \geq 0, \qquad t \in \mathbb{N}_{\boldsymbol{a-1}}$$

Then $y(t)$ is α-increasing.

Proof.

$$({}^{CFR}\nabla_{a-1}^{\alpha} y)(t) = B(\alpha)\nabla_t \sum_{s=a}^{t} y(s)(1-\alpha)^{t-s}$$

$$= B(\alpha)[\sum_{s=a}^{t} y(s)(1-\alpha)^{t-s} - \sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s-1}]$$

$$= B(\alpha)[\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s} + y(t) - \sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s-1}]$$

$$= B(\alpha)[y(t) + \sum_{s=a}^{t-1} y(s)((1-\alpha)^{t-s} - (1-\alpha)^{t-s-1})]$$

$$= B(\alpha)[y(t) + \sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s-1}(1-\alpha-1)]$$

$$= B(\alpha)[y(t) - \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s}] \quad (2.1)$$

But given that $({}^{CFR}\nabla_{a-1}^{\alpha} y)(t) \geq 0$, we have

$$B(\alpha)\left[y(t) - \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s}\right] \geq 0.$$

And since $(\alpha) \geq 0$, we get

$$y(t) - \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s} \geq 0.$$

It follows

$$y(t) \geq \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s}. \quad (2.2)$$

Putting $t = a$ for (8) we get $(a) \geq 0$, put $t = a + 1$ for into (8) , we get

$$y(a+1) \geq \frac{\alpha}{1-\alpha} y(a)(1-\alpha).$$

It follows

$$y(a+1) \geq \alpha y(a).$$

And hence $y(a+1) \geq \alpha y(a) \geq 0$, we will proceed by induction, we get $y(a+k) \geq 0$, for all $k \in \mathbb{N}_{\mathbf{0}}$ which is the same with $y(t) \geq 0$ for all $t \in \mathbb{N}_{\boldsymbol{a}}$.

Now replacing t with $t+1$ in (8) we get

$$y(t+1) \geq \frac{\alpha}{1-\alpha} \sum_{s=a}^{t} y(s)(1-\alpha)^{t-s+1}.$$

Also, we have

$$y(t+1) \geq \alpha y(t) + \frac{\alpha}{1-\alpha} \sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s+1}.$$

And since $\alpha \in (0,1)$ and $y(t) \geq 0$ for all $\in \mathbb{N}_{\boldsymbol{a}}$, we can write

$$y(t+1) \geq \alpha y(t) + \frac{\alpha}{1-\alpha} \sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s+1} \geq \alpha y(t)$$

$$y(t+1) \geq \alpha y(t)$$

which completes the proof.

2.1.2. Theorem

Let y be a function defined on $\mathbb{N}_{\boldsymbol{a-1}}$, if

$\alpha \in (0,1)$, and

$$({}^{CFC}\nabla_{a-1}^{\alpha} y)(t) \geq -\frac{B(\alpha)}{1-\alpha} y(a-1)(1-\alpha)^{t-a+1}, \qquad t \in \mathbb{N}_{\boldsymbol{a-1}}.$$

Then y(t) is α-increasing.

Proof. By assumption, we have

$$({}^{CFC}\nabla_{a-1}^{\alpha}y)(t)+\frac{B(\alpha)}{1-\alpha}y(a-1)(1-\alpha)^{t-a+1}\geq 0$$

and from proposition 1.1.1. we get

$$({}^{CFR}\nabla_{a-1}^{\alpha}y)(t)\geq 0, \qquad t\in\mathbb{N}_{\boldsymbol{a-1}}$$

and from theorem 2.1.1. we get

$y(t)$ is α-increasing, hence the proof complete.

2.1.3. Theorem

Let y be a function defined on $\mathbb{N}_{\boldsymbol{a-1}}$ satisfying $y(a)\geq 0$ and increasing on $\mathbb{N}_{\boldsymbol{a}}$. Then, for $\alpha\in(0,1)$

$$({}^{CFR}\nabla_{a-1}^{\alpha}y)(t)\geq 0, \qquad t\in\mathbb{N}_{\boldsymbol{a-1}}.$$

Proof. From (7), we have

$$({}^{CFR}\nabla_{a-1}^{\alpha}y)(t)=B(\alpha)[y(t)-\frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1}y(s)(1-\alpha)^{t-s}]$$

and since $B(\alpha)\geq 0$ so to show that $({}^{CFR}\nabla_{a-1}^{\alpha}y)(t)\geq 0$ we need to prove that

$y(t)-\frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1}y(s)(1-\alpha)^{t-s}\geq 0$

$$y(t)-\frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1}y(s)(1-\alpha)^{t-s}$$

$$=y(t)-\alpha y(t-1)-\frac{\alpha}{1-\alpha}\sum_{s=a}^{t-2}y(s)(1-\alpha)^{t-s}$$

$$= y(t) - \alpha y(t-1)$$

$$- \frac{\alpha}{1-\alpha} \sum_{s=a}^{t-2} [y(s) - y(t-1) + y(t-1)](1-\alpha)^{t-s}$$

$$= y(t) - \alpha y(t-1)$$

$$- \frac{\alpha}{1-\alpha} \left[\sum_{s=a}^{t-2} \big(y(s) - y(t-1)\big)(1-\alpha)^{t-s} + \sum_{s=a}^{t-2} y(t-1)(1-\alpha)^{t-s} \right], \qquad (2.3)$$

Since y is increasing, it indicates that $y(t) \geq y(t-1) \geq$

$y(t-2) \geq \cdots \geq y(a) \geq 0$, so we get

$$\geq y(t) - \alpha y(t-1) - \frac{\alpha}{1-\alpha} \sum_{s=a}^{t-2} y(t-1)(1-\alpha)^{t-s}$$

$$= y(t) - \frac{\alpha}{1-\alpha} \sum_{s=a}^{t-1} y(t-1)(1-\alpha)^{t-s}$$

$$= y(t) - y(t-1) + y(t-1) - \frac{\alpha}{1-\alpha} y(t-1) \sum_{s=a}^{t-1} (1-\alpha)^{t-s}$$

$$\geq y(t-1) - \frac{\alpha}{1-\alpha} y(t-1) \sum_{s=a}^{t-1} (1-\alpha)^{t-s}$$

$$= y(t-1) \left[1 - \frac{\alpha}{1-\alpha} \sum_{s=a}^{t-1} (1-\alpha)^{t-s} \right]$$

$$= y(t-1) \left[1 - \frac{\alpha}{1-\alpha} \sum_{s=0}^{t-1-a} (1-\alpha)^{t-(s+a)} \right]$$

$$= y(t-1) \left[1 - \frac{\alpha}{1-\alpha} \sum_{s=0}^{(t-a)-1} (1-\alpha)^{t-a-s} \right]$$

$$= y(t-1) \left[1 - \frac{\alpha}{1-\alpha} (1-\alpha)^{t-a} \sum_{s=0}^{(t-a)-1} \left(\frac{1}{1-\alpha} \right)^{s} \right].$$

By using geometric series, we have

$$= y(t-1)\left[1 - \frac{\alpha}{1-\alpha}(1-\alpha)^{t-a}\left(\frac{1-(\frac{1}{1-\alpha})^{t-a}}{1-\frac{1}{1-\alpha}}\right)\right]$$
$$= y(t-1)\left[1 - \frac{\alpha}{1-\alpha}(1-\alpha)^{t-a}\left(\frac{1-(1-\alpha)^{a-t}}{\frac{-\alpha}{1-\alpha}}\right)\right]$$
$$= y(t-1)[1-(1-\alpha)^{t-a}((1-\alpha)^{a-t}-1)]$$
$$= y(t-1)[1-(1-(1-\alpha)^{t-a})]$$
$$= y(t-1)(1-\alpha)^{t-a} \geq 0, \tag{2.4}$$

which completes the proof.

2.1.4. Theorem

Let y be a function defined on $\mathbb{N}_{\boldsymbol{a-1}}$ satisfy $y(a) \geq 0$ and be strictly increasing on $\mathbb{N}_{\boldsymbol{a}}$. Then, for $\alpha \in (0,1)$

$$({}^{CFR}\nabla^{\alpha}_{a-1}y)(t) > 0, \qquad t \in \mathbb{N}_{\boldsymbol{a-1}}$$

Proof. Similar to the previous theorem, this one can be proven.

2.2 $\boldsymbol{\alpha}$-DECREASE

Let y be a function defined on $\mathbb{N}_{\boldsymbol{a}}$,so that satisfying $y(a) \geq 0$. Then y is named an α-decreasing function on $\mathbb{N}_{\boldsymbol{a}}$ if [13]

$$y(t+1) \leq \alpha y(t) \quad for\ all\ t \in \mathbb{N}_{\boldsymbol{a}}.$$

2.2.1. Theorem

Let y be a function defined on $\mathbb{N}_{\boldsymbol{a-1}}$, if

$\alpha \in (0,1)$, and

$$({}^{CFR}\nabla^{\alpha}_{a-1}y)(t) \leq 0, \qquad t \in \mathbb{N}_{\boldsymbol{a-1}}.$$

Then $y(t)$ is α-decreasing.

Proof. From (7) we have

$$({}^{CFR}\nabla^{\alpha}_{a-1}y)(t) \ = B(\alpha)\left[y(t) - \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s}\right].$$

But given that $({}^{CFR}\nabla^{\alpha}_{a-1}y)(t) \leq 0$, so we have

$$B(\alpha)\left[y(t) - \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s}\right] \leq 0.$$

Since $(\alpha) \geq 0$, we get

$$y(t) - \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s} \leq 0.$$

It follows

$$y(t) \leq \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s}. \tag{2.5}$$

Putting $t = a$ for (11) we get $(a) \leq 0$, put $t = a + 1$ for into (11) , we get

$$y(a+1) \leq \frac{\alpha}{1-\alpha} y(a)(1-\alpha).$$

It follows

$$y(a+1) \leq \alpha y(a)$$

and hence $y(a+1) \le \alpha y(a) \le 0$,we will proceed by induction.We get $y(a+k) \le 0$, for all $k \in \mathbb{N}_{\mathbf{0}}$ which is the same with $y(t) \le 0$ for all $t \in \mathbb{N}_{\boldsymbol{a}}$.

Now replacing t with $t+1$ in (11) we get

$$y(t+1) \le \frac{\alpha}{1-\alpha} \sum_{s=a}^{t} y(s)(1-\alpha)^{t-s+1}.$$

Also, we have

$$y(t+1) \le \alpha y(t) + \frac{\alpha}{1-\alpha} \sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s+1}.$$

And since $\alpha \in (0,1)$ and $y(t) \le 0$ for all $t \in \mathbb{N}_{\boldsymbol{a}}$ so we can write

$$y(t+1) \le \alpha y(t) + \frac{\alpha}{1-\alpha} \sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s+1} \le \alpha y(t)$$

$$y(t+1) \le \alpha y(t).$$

Which completes the prove.

2.2.2. Theorem

Let y be a function defined on $\mathbb{N}_{\boldsymbol{a-1}}$ satisfy $y(a) \ge 0$ and be decreasing on $\mathbb{N}_{\boldsymbol{a}}$. Then, for $\alpha \in (0,1)$

$$({}^{CFR}\nabla_{a-1}^{\alpha} y)(t) \le 0, \qquad t \in \mathbb{N}_{\boldsymbol{a-1}}.$$

Proof. From (7) we have

$$({}^{CFR}\nabla_{a-1}^{\alpha}y)(t) = B(\alpha)[y(t) - \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s}]$$

and since $B(\alpha) \geq 0$ so to show that $({}^{CFR}\nabla_{a-1}^{\alpha}y)(t) \leq 0$ we need to prove that $y(t) - \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s} \leq 0$, now from (9) we have

$$y(t) - \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-1} y(s)(1-\alpha)^{t-s}$$
$$= y(t) - \alpha y(t-1)$$
$$- \frac{\alpha}{1-\alpha}\left[\sum_{s=a}^{t-2}\big(y(s) - y(t-1)\big)(1-\alpha)^{t-s} + \sum_{s=a}^{t-2} y(t-1)(1-\alpha)^{t-s}\right].$$

Since y is increasing, it indicates that $y(t) \leq y(t-1) \leq$

$y(t-2) \leq \cdots \leq y(a) \leq 0$, so we get

$$\leq y(t) - \alpha y(t-1) - \frac{\alpha}{1-\alpha}\sum_{s=a}^{t-2} y(t-1)(1-\alpha)^{t-s}.$$

From (10) we have

$$= y(t-1)(1-\alpha)^{t-a} \leq 0.$$

which completes the proof.

2.3. APPLICATION

We know that $({}^{CF}\nabla_{a}^{-\alpha}\ {}^{CFR}\nabla_{a}^{\alpha}y)(t) = y(t)$. Nevertheless, the following result, delivers an initial condition y(a), will be a instrument to prove our fractional difference mean value theorem.

2.3.1. Theorem

For $\in (0,1)$, we have

$$({}^{CF}\nabla_a^{-\alpha} \quad {}^{CFR}\nabla_{a-1}^{\alpha} y)(t) = y(t) - \alpha y(a). \tag{2.6}$$

Proof. From definition, we have

$$({}^{CF}\nabla_a^{-\alpha} \quad {}^{CFR}\nabla_{a-1}^{\alpha} y)(t) = \quad {}^{CF}\nabla_a^{-\alpha}\left[B(\alpha)\nabla_t \sum_{s=a}^{t} y(s)(1-\alpha)^{t-s}\right]$$
$$= B(\alpha)^{CF}\nabla_a^{-\alpha}\nabla_t\left[y(a)(1-\alpha)^{t-a} + \sum_{s=a+1}^{t} y(s)(1-\alpha)^{t-s}\right]$$
$$= \left[B(\alpha)^{CF}\nabla_a^{-\alpha}\nabla_t y(a)(1-\alpha)^{t-a} + \quad {}^{CF}\nabla_a^{-\alpha} B(\alpha)\nabla_t \sum_{s=a+1}^{t} y(s)(1-\alpha)^{t-s}\right]$$
$$= B(\alpha)^{CF}\nabla_a^{-\alpha}\nabla_t y(a)(1-\alpha)^{t-a} + \quad {}^{CF}\nabla_a^{-\alpha} \quad {}^{CFR}\nabla_a^{\alpha} y)(t)$$
$$= B(\alpha) y(a)(1-\alpha)^{-a\,CF}\nabla_a^{-\alpha}\nabla_t (1-\alpha)^{t} + y(t).$$

From lemma (iv) we have

$$= -\alpha B(\alpha) y(a)(1-\alpha)^{-a\,CF}\nabla_a^{-\alpha}(1-\alpha)^{t-1} + y(t).$$

and from Lemma (i) also we get

$$= -\alpha B(\alpha) y(a)(1-\alpha)^{-a}\frac{(1-\alpha)^{a}}{B(\alpha)} + y(t)$$
$$= y(t) - \alpha y(a).$$

The proofs complete.

2.3.2. Theorem

Let f and g be functions defined on $\mathbb{N}_{\boldsymbol{a},\boldsymbol{b}} = \{a, a+1, \dots, b-1, b\}$ where $a < b$ with $a \equiv b\ (mod\ 1)$. Suppose that g is strictly increasing and $\alpha \in (0,1)$. Then, $\exists\ s_1 s_2 \in \mathbb{N}_{\boldsymbol{a},\boldsymbol{b}}$ such that

$$\frac{({}^{CFR}\nabla^{\alpha}_{a-1}f)(s_1)}{({}^{CFR}\nabla^{\alpha}_{a-1}g)(s_1)} \leq \frac{f(b) - \alpha f(a)}{g(b) - \alpha g(a)} \leq \frac{({}^{CFR}\nabla^{\alpha}_{a-1}f)(s_2)}{({}^{CFR}\nabla^{\alpha}_{a-1}g)(s_2)}. \tag{2.7}$$

Proof. We employ contradiction, letting (13) be untrue either then

$$\frac{f(b) - \alpha f(a)}{g(b) - \alpha g(a)} > \frac{({}^{CFR}\nabla^{\alpha}_{a-1}f)(t)}{({}^{CFR}\nabla^{\alpha}_{a-1}g)(t)}\ , for\ all\ \ t \in \mathbb{N}_{\boldsymbol{a},\boldsymbol{b}}, \tag{2.8}$$

Or

$$\frac{f(b) - \alpha f(a)}{g(b) - \alpha g(a)} < \frac{({}^{CFR}\nabla^{\alpha}_{a-1}f)(t)}{({}^{CFR}\nabla^{\alpha}_{a-1}g)(t)}, for\ all\ \ t \in \mathbb{N}_{\boldsymbol{a},\boldsymbol{b}}. \tag{2.9}$$

Given that g is strictly increasing, so by Theorem 2.1.4 we have $({}^{CFR}\nabla^{\alpha}_{a-1}g)(t) > 0$, hence from (14) we get

$$\frac{f(b) - \alpha f(a)}{g(b) - \alpha g(a)} ({}^{CFR}\nabla^{\alpha}_{a-1}g)(t) > ({}^{CFR}\nabla^{\alpha}_{a-1}f)(t).$$

Now take the fractional sum for both sides it becomes

$$\frac{f(b) - \alpha f(a)}{g(b) - \alpha g(a)} ({}^{CF}\nabla^{-\alpha}_{a}\quad {}^{CFR}\nabla^{\alpha}_{a-1}g)(t) > ({}^{CF}\nabla^{-\alpha}_{a}\quad {}^{CFR}\nabla^{\alpha}_{a-1}f)(t).$$

And from (12) we have

$$\frac{f(b) - \alpha f(a)}{g(b) - \alpha g(a)} g(t) - \alpha g(a) > f(t) - \alpha f(a).$$

By t=b substitution

$$f(b) - \alpha f(a) > f(b) - \alpha f(a)$$

which is in contradiction, and we can show that (15) can also result in contradiction. The proof is complete.

PART 3

CONCLUSION

This thesis presents some new alpha-monotonicity analysis results for discrete Caputo-Fabrizio fractional differences in the sense of Riemann-Liouville and Caputo operators. The monotonicity of the function (increasing or decreasing) has been obtained from the positivity or negativity of the discrete Caputo-Fabrizio fractional operator. As a result, we provide the connection between the Riemann-Liouville and Caputo senses of the operators so that we may get the relevant conclusions using Caputo operators. In addition, a discrete mean value theorem is given to show the established results.

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RESUME

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