CONVEXITY AND MONOTONICITY ANALYSES FOR DISCRETE FRACTIONAL OPERATORS WITH DISCRETE EXPONENTIAL KERNELS

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I declare that this project (Convexity and Monotonicity Analyses for Discrete Fractional Operators with Discrete Exponential Kernels) is the result of my own work except as cited in the references. The project has not been accepted for any degree and is not concurrently submitted in candidature of any other degree."

Ihsan Hasan SAADULLAH

ABSTRACT

Master Thesis

CONVEXITY AND MONOTONICITY ANALYSES FOR DISCRETE FRACTIONAL OPERATORS WITH DISCRETE EXPONENTIAL KERNELS

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For discrete fractional operators with exponential kernels, positivity, monotonicity, and convexity findings were taken into consideration in this thesis. Our findings cover both sequential and non-sequential scenarios and show how fractional differences with other kinds of kernels and the exponential kernel example are comparable and different. This demonstrates that the qualitative information gathered in the exponential kernel case does not match other situations perfectly

Keywords **:** Discrete Fractional Calculus, Exponential Kernel, Positivity Analysis, Monotonicity Analysis, Convexity Analysis.
Science Code : 20406

ÖZET

Yüksek Lisans Tezi

AYRIK ÜSTEL ÇEKİRDEKLERE SAHIP AYRIK KESİRLİ OPERATÖRLER İÇİN KONVEKSLİK VE MONOTONLUK ANALİZLERİ

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Bu tezde üstel çekirdekli ayrık kesirli operatörler için pozitiflik, monotonluk ve konvekslik bulguları incelendi. Bulgularımız hem sıralı hem de sıralı olmayan durumlari kapsar ve diğer çekirdek türleri ve üstel çekirdek örneği ile kesirli farklılıkların nasıl karşılaştırılabildiğini ve farklı olduğunu gösterir. Bu, üstel çekirdek durumunda toplanan nitel bilgilerin diğer durumlarla tam olarak eşleşmediğini gösterir.

Anahtar Sözcükler : Ayrık kesirli hesap, üstel çekirdek, pozitiflik analizi, monotonluk analizi, dışbükeylik analizi.

Bilim Kodu **:** 20406

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CONTENTS

Page

APPROVAL ii
ABSTRACT iv
ÖZET v
ACKNOWLEDGEMEENTS vi
CONTENTS vii
LIST OF ABBREVIATIONS ix

PART 1 1
INTRODUCTION 1
1.1. PRELIMINARIES 3
1.1.1. Backward Difference 3
1.1.2. Caputo Fractional Difference 4
1.1.3. Higher Order Fractional Difference 4

CHAPTER TWO 5
MONOTONICITY AND CONVEXITY 5
2.1. $\boldsymbol{\alpha}$-MONOTONICITY 5
2.1.1. $\boldsymbol{\alpha}$-Monotone Increasing 5
2.1.1. Lemma 5
2.1.2. Theorem 8
2.1.3. THEOREM 12
2.2. MONOTONICITY $\boldsymbol{\alpha}$-CONVEXITY 12
2.2.1. Lemma 12
2.2.2. Lemma 14
2.2.3. Lemma 16
2.2.4. Lemma 17
2.2.1. $\boldsymbol{\alpha}$- Convex 18
2.2.5. Lemma 19
2.3. CONVEXITY 20
2.3.1. LEMMA 20

Page

PART 3 22
CONCLUSION 22

REFERENCE 23

RESUME 25

LIST OF ABBREVIATIONS

SYMBOLS

$\Gamma(\alpha)$	: Gamma function.
∇	: Backward (nabla) difference operator.
∇_a^α	: Backward (nabla) difference operator of order α with start point a.
${}^{CFC}\nabla_a^\alpha$	: Caputo-Fabrizio in the Caputo sense
$\rho(s)$	: Backward jumping operator.

PART 1

INTRODUCTION

Numerous scholars from various disciplines, including mathematics, biology, physics, chemistry, engineering, even economics and social sciences, have been concentrating on the discrete fractional calculus field in recent years [1], [2], [3], [4]. A crucial effort has been made, particularly in the field of viscoelasticity, to use fractional mathematical models to better accurately describe the behavior of materials.

In mathematics, the idea of monotonicity is crucial. Unfortunately, there are no monotonicity findings for fractional operators in the theory or applications of fractional calculus. The discrete fractional operators underwent a monotonicity study that was started by Dahal and Goodrich in [5] and Goodrich in [6]. For fractional orders between 0 and 1, however, monotonicity concerns were not taken into consideration. Since non-integer orders were the main focus of the first section of this thesis, we were able to announce new definitions of monotonicity perceptions. Indicators of the mechanical properties of biomaterials are frequently linear differential equations created from physical spring and dashpot models. However, it has been shown that biological tissues exhibit more complicated performance, such as hysteresis, fatigue, and memory, which cannot be explained by combining perfect spring and dashpot combinations [2]. Since the tissues in the human body are naturally viscoelastic, it is important to incorporate correct viscoelastic when studying the mechanics of deformation [7]. The mechanical properties of living soft tissues create a unique combination of testing and modeling problems. To construct stress-strain correlations for viscoelastic materials, fractional calculus is employed.

It is acknowledged that the description of the characteristics of viscoelastic materials has long relied heavily on rheological constitutive equations with fractional derivatives [4]. First-order derivatives in the rheological constitutive equations must be replaced by fractional order derivatives. They are ideal for describing things with memory, such polymers or tissues, as the fractional derivative of a function depends on its whole history rather than on its instantaneous behavior [8]. We created discrete fractional rheological models for the reasons listed above. A material is described by a finite number of springs and dashpots in discrete models.

Although, there are now many different approaches to show a fractional sum and difference, they all have the important trait of being non-local. For instance, Riemann-Liouville definition, one of the most prominent fractional differences, states that the following approach is presented in the case of a backward or nabla difference:

$$(\nabla_a^v f)(t) = \sum_{s=a+1}^{t} H_{-v-1}\left(t, \rho(s)\right) f(s) \tag{1.1}$$

for each $t \in \mathbb{N}_{a+N}$ where

$$H_\mu(t,a) = \frac{\Gamma(t-a+\mu)}{\Gamma(\mathrm{t-a})\Gamma(\mu+1)} \quad and \quad \rho(s) = s-1.$$

The properties that we consider in this thesis is happen in two management. Non-sequential which is single fractional difference operation for instance:

$$(\nabla_a^v f)(t) \geq 0.$$

And sequential which is composition of fractional deference operators such as:

$$\left(\nabla_{a+1}^v \nabla_a^\mu f\right)(t),$$

The following thesis, it has been broadening this research to discrete fractional operators with exponential kernels.

We do not need to impose these kinds of limitations on the parameters' locations based on the findings of this thesis. Our qualitative findings, in particular, do not significantly differ from one regime to the next. Instead, they are valid over the complete range of allowed (μ,ν) parameters. This shows that, somewhat surprisingly, the results we may get for a discrete fractional operator with an exponential kernel are different from those for a discrete fractional operator with a Riemann-Liouville kernel.

We discuss the general structure of the remaining theses before we finish. First, we quickly go through the prerequisites for the remaining theses. The link between the sign of a suitable Caputo-Fabrizio fractional difference in the Caputo sense and, respectively, the positivity, monotonicity, and convexity of the function on which the difference works, is then discussed.

1.1. PRELIMINARIES

It will be most important to our conclusions in the next sections to start by recalling a few basic results from the difference calculus. It is noted that in every part of this thesis standard convention was followed for instance that $\sum_{k=m}^{n} a_k = 0$ whenever $n < m$ [9]. Moreover, represent by $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ for each $\in \mathbb{R}$. The readers are advised to consult the sources [9],[10] for further details on both the discrete fractional calculus and the nabla difference calculus.

1.1.1. Backward Difference

Let $u\colon \mathbb{N}_a \to \mathbb{R}$, the first order backward (nabla) difference of u is defined as follows [9]:

$$(\nabla u)(t) = u(t) - u(t-1), \qquad t \in \mathbb{N}_{a+1},$$

and by using the following notation, we defined the N^{th}-order nabla difference of u:

$$(\nabla^N y)(t) = \big(\nabla(\nabla^{N-1} u)\big)(t), \qquad t \in \mathbb{N}_{a+N},$$

where $N \in \mathbb{N}_1$.

1.1.2. Caputo Fractional Difference

For the function u define on $\mathbb{N}_a$ and α be between 0 and 1, the α^{th}-order Caputo-Fabrizio in the Caputo sense nabla difference of u is introduced by [11]:

$$({}^{CFC}\nabla_a^\alpha u)(t) = B(\alpha)\sum_{s=a+1}^{t}(\nabla u)(s)(1-\alpha)^{t-s}, t \in \mathbb{N}_{a+1},$$

and the function $\alpha \rightarrow B(\alpha)$ is a normalization constant with $B(0) = B(1) = 1$ and $B(\alpha) > 0$.

1.1.3. Higher Order Fractional Difference

Let $n \leq \alpha \leq n+1$ and u define on $\mathbb{N}_{a-n}$. The α-order Caputo-Fabrizio in the Caputo sense of u is given by [12]:

$$({}^{CFC}\nabla_a^\alpha u)(t) = ({}^{CFC}\nabla_a^{\alpha-n}\nabla^n u)(t), \qquad t \in \mathbb{N}_{a+1}.$$

CHAPTER TWO

MONOTONICITY AND CONVEXITY

2.1. α-MONOTONICITY

In this section several results have been proved, which establish a linking between the sign of suitable Caputo-Fabrizio operator in the Caputo sense fractional nabla difference and the function's positivity on which it performances. Moreover, it is also deduced some results regarding what it is termed, a perception which was given in an article that Goodrich and Lizama recently published [13]. It starts out by defining α-monotone increasing.

2.1.1. α-Monotone Increasing

Let α be between 0 and 1 the function u define on $\mathbb{N}_a$, is called α-monotone increasing if

$$u(t) \geq \alpha u(t-1), t \in \mathbb{N}_{a+1}.$$

Note that in the above definition if $\alpha = 1$, then it is obtained $u(t) \geq u(t-1)$ for $t \in \mathbb{N}_{a+1}$, which is 1-monotone increasing and represents monotonicity in the usual sense. And if $\alpha = 0$, then it is acquired $u(t) \geq 0$, for $t \in \mathbb{N}_{a+1}$. So, it's shows that for us 0-monotone increasing merely indicate that u is nonnegative on $\mathbb{N}_a$.

2.1.1. Lemma

Let the function u is defined on $\mathbb{N}_a$ and $\alpha \in (0,1)$. If

$$({}^{CFC}\nabla_a^{\alpha} u)(t) \geq 0, \qquad for\ all\ t \in \mathbb{N}_{a+1},$$

and $u(a) \geq 0$, then u is positive and α-monotone increasing on $\mathbb{N}_a$.

Proof.

$$\begin{aligned}
&({}^{CFC}\nabla_a^{\alpha}u)(t)\\
&= B(\alpha)\sum_{s=a+1}^{t}(\nabla u)(s)(1-\alpha)^{t-s}\\
&= B(\alpha)\left[\sum_{s=a+1}^{t}[u(s)-u(s-1)](1-\alpha)^{t-s}\right]\\
&= B(\alpha)\left[\sum_{s=a+1}^{t}u(s)(1-\alpha)^{t-s}-\sum_{s=a+1}^{t}u(s-1)(1-\alpha)^{t-s}\right]\\
&= B(\alpha)\left[\sum_{s=a+1}^{t}u(s)(1-\alpha)^{t-s}-\sum_{s=a}^{t-1}u(s)(1-\alpha)^{t-s-1}\right]\\
&= B(\alpha)\left[\sum_{s=a+1}^{t-1}u(s)(1-\alpha)^{t-s}+u(t)(1-\alpha)^{t-t}-\sum_{s=a+1}^{t-1}u(s)(1-\alpha)^{t-s-1}\right.\\
&\qquad\left.-u(a)(1-\alpha)^{t-a-1}\right]\\
&= B(\alpha)\left[\sum_{s=a+1}^{t-1}u(s)(1-\alpha)^{t-s}+u(t)(1-\alpha)^{0}-\sum_{s=a+1}^{t-1}u(s)(1-\alpha)^{t-s-1}\right.\\
&\qquad\left.-u(a)(1-\alpha)^{t-a-1}\right]\\
&= B(\alpha)\left[u(t)-u(a)(1-\alpha)^{t-a-1}+\sum_{s=a+1}^{t-1}u(s)(1-\alpha)^{t-s}\right.\\
&\qquad\left.-\sum_{s=a+1}^{t-1}\frac{u(s)(1-\alpha)^{t-s}}{(1-\alpha)}\right]\\
&= B(\alpha)\left[u(t)-u(a)(1-\alpha)^{t-a-1}+\sum_{s=a+1}^{t-1}u(s)(1-\alpha)^{t-s}\quad(1-\frac{1}{1-\alpha})\right]
\end{aligned}$$

$$= B(\alpha)\left[u(t) - u(a)(1-\alpha)^{t-a-1} - \frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-1} u(s)(1-\alpha)^{t-s}\right] \quad (2.1)$$

But $({}^{CFC}\nabla_a^{\alpha} u)(t) \geq 0,\ for\ all\ t \in \mathbb{N}_{a+1}$, so we can write:

$$B(\alpha)\left[u(t) - u(a)(1-\alpha)^{t-a-1} - \frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-1} u(s)(1-\alpha)^{t-s}\right] \geq 0 \ ,$$

$t \in \mathbb{N}_{a+1}$

It's show that:

$$u(t) \geq u(a)(1-\alpha)^{t-a-1} + \frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-1} u(s)(1-\alpha)^{t-s} \ ,$$

$t \in \mathbb{N}_{a+1}$ (2.2)

Now, to show that u is positive, it's enough to show that $u(a+k) \geq 0$ for any $k \in \mathbb{N}_0$, so we can use induction on k.

Taking $\mathrm{k} = 1$, which is $\mathrm{t} = \mathrm{a} + 1$ in (2.2), and by setting $u(a) \geq 0$, we get:

$$u(a+1) \geq u(a) \geq 0$$

Taking $k = 2$, which is $t = a + 2$ *in (3.2)* , *we have:*

$$u(a+2) \geq u(a)(1-\alpha)^{a+2-a-1} + \frac{\alpha}{1-\alpha}\sum_{s=a+1}^{a+2-1} u(s)(1-\alpha)^{a+2-s} .$$

If we make it simpler*:*

$$u(a+2) \geq u(a)(1-\alpha) + \frac{\alpha}{1-\alpha} \sum_{s=a+1}^{a+1} u(s)(1-\alpha)^{a+2-s} \;,$$

and by setting $u(a) \geq 0$, $u(a+1) \geq 0$ and $0 < \alpha < 1$, we get

$$u(a+2) \geq u(a)(1-\alpha) + \alpha u(a+1) \geq 0.$$

From induction it is acquired $u(t) \geq 0$, $\forall\, t \in \mathbb{N}_a$.

Now to demonstrate that u is α-monotone increasing on $\mathbb{N}_a$. Arrange differently the terms in (2.2), we will get

$$u(t) \geq \alpha u(t-1) + u(a)(1-\alpha)^{t-a-1} + \frac{\alpha}{1-\alpha} \sum_{s=a+1}^{t-2} u(s)(1-\alpha)^{t-s},$$

$$t \in \mathbb{N}_{a+1}. \tag{2.3}$$

since $u(t) \geq 0$, for all $t \in \mathbb{N}_{a+1}$, this comes from (2.3) that

$$u(t) \geq \alpha u(t-1), \qquad t \in \mathbb{N}_{a+1},$$

By getting that u is α -monotone increasing on $\mathbb{N}_a$. The proof is complete.

2.1.2. Theorem

Let the function u is defined on $\mathbb{N}_a$ *and* $\alpha\,,\beta \in (0,1)$, such that $0 < \alpha + \beta \leq 1$ *If*

$$({}^{CFC}\nabla_{a+1}^{\beta} \quad {}^{CFC}\nabla_a^{\alpha} u)(t) \geq 0, \qquad for\ all\ t \in \mathbb{N}_{a+2},$$

and $u(a+1) \geq u(a) \geq 0$, *then u is positive and* $\alpha + \beta$*-monotone increasing on* $\mathbb{N}_a$.

Proof. Let

$$({}^{CFC}\nabla_a^{\alpha}u)(t) = f(t), \qquad for\ all\ t \in \mathbb{N}_{a+1}.$$

So we can write

$$({}^{CFC}\nabla_{a+1}^{\beta}\ {}^{CFC}\nabla_a^{\alpha}u)(t) = ({}^{CFC}\nabla_{a+1}^{\beta}f)(t).$$

Distinctly, by assumption, $({}^{CFC}\nabla_{a+1}^{\beta}f)(t),\ for\ all\ t \in \mathbb{N}_{a+2}$. By definition of Caputo fractional difference we possess:

$$f(a+1) = ({}^{CFC}\nabla_a^{\alpha}u)(a+1) = B(\alpha)\sum_{s=a+1}^{a+1}(\nabla u)(s)(1-\alpha)^{a+1-s}$$
$$= B(\alpha)(\nabla u)(a+1)(1-\alpha)^{a+1-(a+1)} = B(\alpha)(\nabla u)(a+1) \geq 0$$

According to Lemma 2.1.1, f is positive and β-monotone increasing on $\mathbb{N}_{a+1}$.

So

$$f(t) \geq 0, \qquad for\ all\ t \in \mathbb{N}_{a+1},$$

$$f(t) \geq \beta f(t-1), \qquad for\ all\ t \in \mathbb{N}_{a+2}, \tag{2.4}$$

since

$$f(t) = ({}^{CFC}\nabla_a^{\alpha}u)(t) \geq 0, \qquad for\ all\ t \in \mathbb{N}_{a+1}.$$

and $u(a) \geq 0$, again from lemma 2.1.1, u is positive and α-monotone increasing on $\mathbb{N}_a$. that is,

$$u(t) \geq 0, \qquad for\ all\ t \in \mathbb{N}_a.$$

And

$$u(t) \geq \alpha u(t-1), \qquad t \in \mathbb{N}_{a+1}, \tag{2.5}$$

Now, from (2.4) it has

$$f(t) \geq \beta f(t-1), \qquad for\ all\ t \in \mathbb{N}_{a+2}.$$

It is shown that

$$0 \leq f(t) - \beta f(t-1) = ({}^{CFC}\nabla_a^{\alpha} u)(t) - \beta({}^{CFC}\nabla_a^{\alpha} u)(t-1)$$

and by applying (2.1) we get:

$$= B(\alpha)\left[u(t) - u(a)(1-\alpha)^{t-a-1} - \frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-1} u(s)(1-\alpha)^{t-s}\right]$$
$$- B(\alpha)\beta\Bigg[u(t-1) - u(a)(1-\alpha)^{t-a-2}$$
$$- \frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-2} u(s)(1-\alpha)^{t-s-1}\Bigg]$$
$$= B(\alpha)\Bigg[u(t) - u(a)(1-\alpha)^{t-a-1} - \frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-2} u(s)(1-\alpha)^{t-s} - \alpha u(t-1)$$
$$- \beta u(t-1) + \beta u(a)(1-\alpha)^{t-a-2}$$
$$+ \beta\frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-2} u(s)(1-\alpha)^{t-s-1}\Bigg]$$
$$= B(\alpha)\Bigg[u(t) - \beta u(t-1) - u(a)(1-\alpha)^{t-a-1} + \beta u(a)(1-\alpha)^{t-a-2}$$
$$- \frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-2} u(s)(1-\alpha)^{t-s} - \alpha u(t-1)$$
$$+ \beta\frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-2} u(s)(1-\alpha)^{t-s-1}\Bigg]$$

$$= B(\alpha)\Bigg[u(t) - \beta u(t-1) - u(a)(1-\alpha)^{t-a-2}(1-\alpha-\beta)$$
$$- \frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-2} u(s)[(1-\alpha)^{t-s} - \beta(1-\alpha)^{t-s-1}] - \alpha u(t-1)\Bigg]$$
$$= B(\alpha)\Bigg[u(t) - \beta u(t-1) - u(a)(1-\alpha)^{t-a-2}(1-\alpha-\beta)$$
$$- \frac{\alpha}{1-\alpha}\sum_{s=a+1}^{t-2} u(s)(1-\alpha)^{t-s-1}(1-\alpha-\beta) - \alpha u(t-1)\Bigg]$$
$$= B(\alpha)\Bigg[u(t) - \beta u(t-1) - \alpha u(t-1) - u(a)(1-\alpha)^{t-a-2}(1-\alpha-\beta)$$
$$- \frac{\alpha(1-\alpha-\beta)}{(1-\alpha)^2}\sum_{s=a+1}^{t-2} u(s)(1-\alpha)^{t-s}\Bigg], \qquad (2.6)$$

Since $B(\alpha) > 0$, from (2.6) it is gained:

$$u(t) - \beta u(t-1) - \alpha u(t-1) - u(a)(1-\alpha)^{t-a-2}(1-\alpha-\beta)$$
$$- \frac{\alpha(1-\alpha-\beta)}{(1-\alpha)^2}\sum_{s=a+1}^{t-2} u(s)(1-\alpha)^{t-s} \geq 0, \quad t \in \mathbb{N}_{a+2}.$$

The following is also considered

$$u(t) - \beta u(t-1) - \alpha u(t-1)$$
$$\geq u(a)(1-\alpha)^{t-a-2}(1-\alpha-\beta)$$
$$+ \frac{\alpha(1-\alpha-\beta)}{(1-\alpha)^2}\sum_{s=a+1}^{t-2} u(s)(1-\alpha)^{t-s},$$
$$t \in \mathbb{N}_{a+2}. \qquad (2.7)$$

Since $0 < \alpha \leq 1, 0 < \alpha + \beta \leq 1$, and $(t) \geq 0$, from (2.7) we can write:

$$u(t) - \beta u(t-1) - \alpha u(t-1) \geq 0, \qquad t \in \mathbb{N}_{a+2}.$$

Also we get:

$$u(t) \geq (\beta + \alpha)u(t-1), \qquad t \in \mathbb{N}_{a+2}.$$

Hence the prove is complete.

2.1.3. THEOREM

Let the function u is defined on $\mathbb{N}_a$ *and* $\alpha, \beta \in (0,1)$, such that $0 < \alpha + \beta \leq 1$. *If*

$$({}^{CFC}\nabla_{a+1}^{\beta} \quad {}^{CFC}\nabla_a^{\alpha} u)(t) \geq 0, \qquad for\ all\ t \in \mathbb{N}_{a+2}.$$

and $u(a+1) \geq u(a) \geq 0$, then u is positive and α-monotone increasing on $\mathbb{N}_a$.

Proof. It is demonstrated by the theorem mentioned above.

2.2. MONOTONICITY $\boldsymbol{\alpha}$-CONVEXITY

Once more, it was during a conference of monotonicity-type results. For both non-sequential and sequential Caputo-Fabrizio, fractional differences are found in the Caputo sense. It begins with two fundamental lemmas.

2.2.1. Lemma

Let the function u is defined on $\mathbb{N}_a$ *and* $\alpha \in (0,1)$. *If*

$$\nabla({}^{CFC}\nabla_a^{\alpha} u)(t) \geq 0, \qquad for\ all\ t \in \mathbb{N}_{a+2},$$

and $(\nabla u)(a+1) \geq 0$, then

$$(\nabla u)(t) \geq 0, \qquad for\ all\ t \in \mathbb{N}_{a+1}.$$

Proof. First, by definition of Caputo fractional difference, for all $t \in \mathbb{N}_{a+2}$ we have

$$\nabla({}^{CFC}\nabla_a^{\alpha}u)(t) = \nabla\left[B(\alpha)\sum_{s=a+1}^{t}(\nabla u)(s)(1-\alpha)^{t-s}\right]$$

$$= B(\alpha)\left[\sum_{s=a+1}^{t}(\nabla u)(s)(1-\alpha)^{t-s} - \sum_{s=a+1}^{t-1}(\nabla u)(s)(1-\alpha)^{t-1-s}\right]$$

$$= B(\alpha)\left[\sum_{s=a+1}^{t-1}(\nabla u)(s)(1-\alpha)^{t-s} + (\nabla u)(t) - \sum_{s=a+1}^{t-1}(\nabla u)(s)(1-\alpha)^{t-1-s}\right]$$

$$= B(\alpha)\left[(\nabla u)(t) + \sum_{s=a+1}^{t-1}(\nabla u)(s)((1-\alpha)^{t-s} - (1-\alpha)^{t-1-s})\right]$$

$$= B(\alpha)\left[(\nabla u)(t) - \frac{\alpha}{\alpha-1}\sum_{s=a+1}^{t-1}(\nabla u)(s)(1-\alpha)^{t-s}\right], \tag{2.8}$$

Since $\nabla({}^{CFC}\nabla_a^{\alpha}u)(t) \geq 0$, we have:

$$B(\alpha)\left[(\nabla u)(t) - \frac{\alpha}{\alpha-1}\sum_{s=a+1}^{t-1}(\nabla u)(s)(1-\alpha)^{t-s}\right] \geq 0$$

and since $B(\alpha) \geq 0$, so it is written:

$$(\nabla u)(t) - \frac{\alpha}{\alpha-1}\sum_{s=a+1}^{t-1}(\nabla u)(s)(1-\alpha)^{t-s} \geq 0.$$

Also it has the same meaning with this:

$$(\nabla u)(t) \geq \frac{\alpha}{\alpha-1}\sum_{s=a+1}^{t-1}(\nabla u)(s)(1-\alpha)^{t-s}. \tag{2.9}$$

Now to show that $(\nabla u)(t) \geq 0$ for all $t \in \mathbb{N}_{a+1}$, its enough to show that $(\nabla u)(a + k) \geq 0$, $\forall\, k \in \mathbb{N}_1$.

The usage of induction on k. $(\nabla u)(a+1) \geq 0$ is given. Taking

$t = a + 2$ in (2.9) , we have:

$$(\nabla u)(a+2) \geq \frac{\alpha}{\alpha - 1} \sum_{s=a+1}^{a+1} (\nabla u)(s)(1-\alpha)^{a+2-s}$$

By simplifying it we get:

$$(\nabla u)(a+2) \geq \alpha(\nabla u)(a+1) \geq 0.$$

Taking $t = a + 3$ in (2.9), it becomes

$$(\nabla u)(a+3) \geq \alpha(\nabla u)(a+2) + (\nabla u)(a+1)(1-\alpha) \geq 0.$$

Following the same procedure, it is obtained $(\nabla u)(t) \geq 0$, for all $t \in \mathbb{N}_{a+1}$. Hence The proof has been established.

2.2.2. Lemma

Let the function u is defined on $\mathbb{N}_a$ *and* $\alpha \in (0,1)$. If

$$({}^{CFC}\nabla_{a+1}^{\alpha} \nabla u)(t) \geq 0, \qquad for\ all\ t \in \mathbb{N}_{a+2},$$

and $(\nabla u)(a+1) \geq 0$, then

$$(\nabla u)(t) \geq 0, \qquad for\ all\ t \in \mathbb{N}_{a+1}.$$

Proof. By definition of Caputo fractional difference, for all $t \in \mathbb{N}_{a+2}$, we have

$$({}^{CFC}\nabla_{a+1}^{\alpha} \nabla u)(t) = B(\alpha) \sum_{s=a+2}^{t} (\nabla^2 u)(s)(1-\alpha)^{t-s}$$

$$= B(\alpha) \left[\sum_{s=a+2}^{t} (\nabla u)(s)(1-\alpha)^{t-s} - \sum_{s=a+2}^{t} (\nabla u)(s-1)(1-\alpha)^{t-s} \right]$$

$$= B(\alpha)\left[\sum_{s=a+2}^{t} (\nabla u)(s)(1-\alpha)^{t-s} - \sum_{s=a+1}^{t-1} (\nabla u)(s)(1-\alpha)^{t-s-1}\right]$$

$$= B(\alpha)\left[\sum_{s=a+2}^{t-1} (\nabla u)(s)(1-\alpha)^{t-s} + (\nabla u)(t) - \sum_{s=a+2}^{t-1} (\nabla u)(s)(1-\alpha)^{t-s-1} \right.$$
$$\left. - (\nabla u)(a+1)(1-\alpha)^{t-a-2}\right]$$

$$= B(\alpha)\left[(\nabla u)(t) + \sum_{s=a+2}^{t-1} (\nabla u)(s)[(1-\alpha)^{t-s} - (1-\alpha)^{t-s-1}] \right.$$
$$\left. - (\nabla u)(a+1)(1-\alpha)^{t-a-2}\right]$$

$$= B(\alpha)\left[(\nabla u)(t) - \frac{\alpha}{1-\alpha}\sum_{s=a+2}^{t-1} (\nabla u)(s)(1-\alpha)^{t-s} \right.$$
$$\left. - (\nabla u)(a+1)(1-\alpha)^{t-a-2}\right], \quad (2.10)$$

nevertheless, because $B(\alpha) \geq 0$, and $({}^{CFC}\nabla^{\alpha}_{a+1}\nabla u)(t) \geq 0,\ for\ all\ t \in \mathbb{N}_{a+2}$, so from (2.10), we get

$$(\nabla u)(t) \geq \frac{\alpha}{1-\alpha}\sum_{s=a+2}^{t-1} (\nabla u)(s)(1-\alpha)^{t-s}$$
$$+ (\nabla u)(a+1)(1-\alpha)^{t-a-2}. \quad (2.11)$$

Now to show that $(\nabla u)(t) \geq 0$ for all $t \in \mathbb{N}_{a+1}$, its enough to show that $(\nabla u)(a+k) \geq 0$, for each $k \in \mathbb{N}_1$.

By applying induction to k. Given that $(\nabla u)(a+1) \geq 0$. Using

$t = a + 2$ in (2.11) , it has got:

$$(\nabla u)(a+2) \geq (\nabla u)(a+1) \geq 0$$

Taking $t = a + 3$ in (2.9), there is

$(\nabla u)(a+3) \geq \alpha(\nabla u)(a+2) + (\nabla u)(a+1)(1-\alpha) \geq 0.$

Continuing in this way, it has reached $(\nabla u)(t) \geq 0$, for all $t \in \mathbb{N}_{a+1}$.

The proof is complete

2.2.3. Lemma

Let the function u is defined on $\mathbb{N}_a$, $\mu \in (0,1)$ *and* $\nu \epsilon (1,2)$. *If*

$$({}^{CFC}\nabla_{a+2}^{\nu} \quad {}^{CFC}\nabla_a^{\mu} u)(t) \geq 0, \qquad for\ all\ t \in \mathbb{N}_{a+3},$$

and $(\nabla u)(a+2) \geq (\nabla u)(a+1) \geq 0$, then

$$(\nabla u)(t) \geq 0, \qquad for\ all\ t \in \mathbb{N}_{a+1}.$$

Proof. Let

$$({}^{CFC}\nabla_a^{\mu} u)(t) = v(t), \qquad for\ all\ t \in \mathbb{N}_{a+1}.$$

Consider

$$({}^{CFC}\nabla_{a+2}^{\nu} v)(t) = ({}^{CFC}\nabla_{a+2}^{\nu-1} \nabla v)(t)$$

and given that $({}^{CFC}\nabla_{a+2}^{\nu-1} \nabla v)(t) \geq 0,\ for\ all\ t \in \mathbb{N}_{a+3}$, from (2.8) we have:

$$\begin{aligned}\nabla v(a+2) = (\nabla^{CFC}\nabla_a^{\mu} u)(a+2) = B(\mu)[(\nabla u)(a+2) \\ -\frac{\mu}{1-\mu}\sum_{s=a+1}^{a+1}(\nabla u)(s)(1-\mu)^{a+2-s}\Bigg] \\ = B(\mu)[(\nabla u)(a+2) - \mu(\nabla u)(a+1)]\end{aligned}$$

$$> B(\mu)[(\nabla u)(a+2) - (\nabla u)(a+1)] \geq 0$$

Then, from lemma 2.2.2, we have:

$$\nabla(v)(t) = (\nabla^{CFC}\nabla^{\mu}_{a+1}u)(t) \geq 0, \tag{2.12}$$

And since $(\nabla u)(a+1) \geq 0$,from lemma 2.2.1, we get

$$\nabla(u)(t) \geq 0, \qquad for\ all\ t \in \mathbb{N}_{a+1}.$$

so the proof is finished.

2.2.4. Lemma

Let the function u is defined on $\mathbb{N}_a$, $\mu \in (1,2)$ *and* $\nu \epsilon (0,1)$. If

$$({}^{CFC}\nabla^{\nu}_{a+2} \quad {}^{CFC}\nabla^{\mu}_{a+1}u)(t) \geq 0, \qquad for\ all\ t \in \mathbb{N}_{a+3},$$

and $(\nabla u)(a+2) \geq (\nabla u)(a+1) \geq 0$, then

$$(\nabla u)(t) \geq 0, \qquad for\ all\ t \in \mathbb{N}_{a+1}.$$

Proof. Let

$$({}^{CFC}\nabla^{\mu}_{a+1}u)(t) = v(t), \qquad for\ all\ t \in \mathbb{N}_{a+1}.$$

Consider

$$({}^{CFC}\nabla^{\nu}_{a+2}v)(t) = ({}^{CFC}\nabla^{\nu-1}_{a+2}v)(t)$$

and given that $({}^{CFC}\nabla^{\nu}_{a+2}v)(t) \geq 0,\ for\ all\ t \in \mathbb{N}_{a+3}$, from (2.10) we have:

$$v(a+2) = ({}^{CFC}\nabla^{\mu}_{a+1}u)(a+2) = ({}^{CFC}\nabla^{\mu-1}_{a+1}\nabla u)(a+2) = B(\mu-1)\Big[(\nabla u)(a+2)$$
$$\left. -\frac{\mu-1}{2-\mu}\sum_{s=a+1}^{a+1}(\nabla u)(s)(2-\mu)^{a+2-s} - (\nabla u)(a+1)(2-\mu)^{a+2-a-2}\right]$$
$$= B(\mu-1)[(\nabla u)(a+2) - \mu(\nabla u)(a+1)]$$
$$> B(\mu-1)[(\nabla u)(a+2) - (\nabla u)(a+1)] \geq 0.$$

Then, from lemma (2.1.1) we have

$$v(t) \geq 0, \qquad for\ all\ t \in \mathbb{N}_{a+2},$$

which is the same with

$$({}^{CFC}\nabla^{\mu}_{a+1}u)(t) \geq 0, \qquad for\ all\ t \in \mathbb{N}_{a+2}, \tag{2.13}$$

Thus, we obtain

$$0 \leq ({}^{CFC}\nabla^{\mu}_{a+1}u)(t) = ({}^{CFC}\nabla^{\mu-1}_{a+1}\nabla u)(t), \qquad for\ all\ t \in \mathbb{N}_{a+2},$$

and since $(\nabla u)(a+1) \geq 0$,from lemma 2.2.2, we get

$$\nabla(u)(t) \geq 0, \qquad for\ all\ t \in \mathbb{N}_{a+1}.$$

so the proof is complete.

2.2.1. α- Convex

Let $\alpha \in (1,2)$ it is mentioned that a function u is defineed on $\mathbb{N}_a$ is called α-convex if

$$u(t) - \alpha u(t-1) + (\alpha-1)u(t-2) \geq 0, \quad t \in \mathbb{N}_{a+2}$$

2.2.5. Lemma

Let the function u is defined on $\mathbb{N}_a$ *and* $\alpha \in (1,2)$. If

$$({}^{CFC}\nabla_{a+1}^{\alpha} u)(t) \geq 0, \qquad for\ all\ t \in \mathbb{N}_{a+2},$$

and

$$u(a+1) \geq u(a) \geq 0.$$

Then u is monotone increasing and positive on $\mathbb{N}_a$. Furthermore, u is α-convex on $\mathbb{N}_a$.

Proof. From the properties we can say

$$0 \leq ({}^{CFC}\nabla_{a+1}^{\alpha} u)(t) = ({}^{CFC}\nabla_{a+1}^{\alpha-1} \nabla u)(t), \qquad for\ all\ t \in \mathbb{N}_{a+2}.$$

And since $\nabla u(a+1) = u(a+1) - u(a) \geq 0$, from lemma (2.2.2) it follows that

$$(\nabla u)(t) \geq 0, \qquad for\ all\ t \in \mathbb{N}_{a+1},$$

which means that u is monotone increasing and positive on $\mathbb{N}_a$.

The next demonstration is that u is α-convex. From (2.11)

$$\begin{aligned}(\nabla u)(t) &\geq \frac{\alpha-1}{2-\alpha}\sum_{s=a+2}^{t-1}(\nabla u)(s)(2-\alpha)^{t-s} + (\nabla u)(a+1)(2-\alpha)^{t-a-2} \\ &= (\alpha-1)(\nabla u)(t-1) + \frac{\alpha-1}{2-\alpha}\sum_{s=a+2}^{t-2}(\nabla u)(s)(2-\alpha)^{t-s} \\ &\quad + (\nabla u)(a+1)(2-\alpha)^{t-a-2}, \qquad (2.14)\end{aligned}$$

Since $\alpha \in (1,2)$ and $(\nabla u)(t) \geq 0$, for all $t \in \mathbb{N}_{a+1}$. from (2.14) it was written

$$(\nabla u)(t) \geq (\alpha-1)(\nabla u)(t-1).$$

If it was simplified it will result the following

$$u(t) - u(t-1) \geq (\alpha - 1)[u(t-1) - u(t-2)].$$

Also it results in:

$$u(t) - \alpha u(t-1) + (\alpha - 1)u(t-2) \geq 0.$$

It is indicated that u has α-convex, hence the proof is completed.

2.3. CONVEXITY

Some convexity-type results were reported in this section. The conclusion is that there is a link between the sign of the non-sequential difference $({}^{CFC}\nabla_{a+2}^{\alpha} u)(t) \geq 0$, and the following lemma, and the convexity of u. This is basically the type of result assumed in Goodrich [6] as well as Jia, Erbe, and Peterson [14].

2.3.1. LEMMA Let $\alpha \in (2,3)$ and the function u is defined on $\mathbb{N}_{\mathrm{a}}$.

If

$$({}^{CFC}\nabla_{a+2}^{\alpha} u)(t) \geq 0, \qquad for\ all\ t \in \mathbb{N}_{a+3}.$$

and

$$(\nabla^2 u)(a+2) \geq 0$$

Then u is convex on $\mathbb{N}_{a+2}$.

Proof. We start with the following substitution

$$(\nabla u)(t) = v(t), \qquad for\ all\ t \in \mathbb{N}_{a+1}.$$

Consider

$$({}^{CFC}\nabla^{\alpha}_{a+2}u)(t) = ({}^{CFC}\nabla^{\alpha-2}_{a+2}\nabla^2 u)(t) = ({}^{CFC}\nabla^{\alpha}_{a+2}\nabla v)(t)$$

Given that $({}^{CFC}\nabla^{\alpha}_{a+2}\nabla v)(t) \geq 0, \ for\ all\ t \in \mathbb{N}_{a+3}$. *Since*

$$(\nabla v)(a+2) = (\nabla^2 u)(a+2) \geq 0.$$

from lemma 2.2.2, it follows that

$$(\nabla v)(t) = (\nabla^2 u)(t) \geq 0.$$

which is

$$\nabla[u(t) - u(t-1)] \geq 0.$$

Also there is

$$[u(t) - u(t-1)] - [u(t-1) - u(t-2)] \geq 0.$$

Now we get

$$u(t) - 2u(t-1) + u(t-2) \geq 0, \qquad for\ all\ t \in \mathbb{N}_{a+2}.$$

The proof is completed.

PART 3

CONCLUSION

In this thesis, it is investigated some positivity, monotonicity, and convexity results for discrete Caputo-Fabrizio fractional operators in the context of discrete fractional calculus. Also, it is considered the connections of these results to the nonnegativity of both non-sequential and sequential Caputo-Fabrizio fractional differences of Caputo type. Finally, it is found that there are some significant dissimilarities between this type of fractional difference and, for instance, the more well-known Riemann-Liouville type.

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RESUME

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