ON SOME NEW DIOPHANTINE EQUATIONS

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Seyran IBRAHIMOV

ABSTRACT

M. Sc. Thesis

ON SOME NEW DIOPHANTINE EQUATIONS

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In this thesis, some new Diophantine equations are introduced. Then, the solutions of the equations in the set of positive integers are found.

First, all solutions to a Pillai-type problem associated with Lucas numbers are determined.

Secondly, the Brocard-Ramanujan equation is solved when the right-hand side is Mersenne numbers.

Finally, it has been shown that some new Fermat-type equations associated with number sequences have no solutions other than trivial solutions.

Key Words : Fermat's last theorem, Diophantine equations, Recurrence number sequences, Pillai problem, Brocard-Ramanujan equation.

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ÖZET

Yüksek Lisans Tezi

BAZI YENİ DIOFANT DENKLEMLERİ ÜZERİNE

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Bu tezde bazı yeni Diofant denklemler tanımlanmış ve bu denklemlerin pozitif tam sayılar kümesindeki çözümleri bulunmuşdur.
İlk olarak, Lucas sayıları ile ilişkili Pillai tipi bir problemin tüm çözümleri bulunmuştur.
Ayrıca, Brocard-Ramanujan denklemi, sağ tarafı Mersenne sayıları olduğu durumda çözülmüştür.
Son olarak, Fermat tipi bazı yeni Diofant denklemlerin aşikâr çözümlerinden başka çözümlerinin olmadığı kanıtlanmıştır.

Anahtar Kelimeler : Fermat'ın son teoremi, Diofant denklemler, Rekürans sayı dizileri, Pillai problemi, Brocard-Ramanujan denklemi.

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SYMBOLS

$\mathbb{N}^*$: set of positive integers

F_n : n -th Fibonacci number

L_n : n-th Lucas number

$\mathbb{Q}$: field of rational integers

$\mathbb{L}$: a real algebraic number field over $\mathbb{Q}$

M_n : n-th Mersenne number

$\mathbb{Z}$: set of integers

PART 1

INTRODUCTION

Diophantine equations are one of the oldest and most curious fields of number theory. If there are two or more unknowns in an equation, this equation is called Diophantine equation. The title of Diophantine is connected with the name of Diophantus, who lived in the *3*rd century. Diophantus made the first studies on these field. Usually, it is necessary to find all the integer solutions of these equations. However, there is no general method with finite numerical calculations for solving these equations in the set of integers. This theorem is the 10th problem among Hilbert's *23* famous problems that Matiyasevich proved in 1970. For this reason, different methods are developed by mathematicians to solve special classes of Diophantine equations. In this work, we will use Baker's reduction technique, the factoring method, and appropriate inequalities to solve our problems.

This thesis is organized as follows:
In section 2, we provide some significant references to ensure the history of our topics. In addition, we deal with the recent works related to our subject.

In section 3, we give the essential definitions, lemmas, and theorems for which it is to prove our results.

In section 4, we give data about the structure of techniques that we used in the proof of our results. In addition, we solved some examples using these methods.

In section 5, we describe our main results and their detailed proofs.

In section 6, we just talk about the summary in detail, important references, and resume with some information.

PART 2

LITERATURE REVIEW

Diophantine equations are one of the absorbing topics of number theory. Some well-known mathematicians such as P. Fermat, P. Erdös, J. L. Lagrange and S. Ramanujan have significant works in this field.

D. Hilbert proposed in *1900*, that there is no method for finding the solution to these equations by finite mathematical operations on the set of integers. Y. Matiyasevich proved this conjecture in *1970* [1]. Also, working on this topic leads to the development of new techniques in number theory. For instance, Fermat's last theorem has led to the development of different fields of number theory.

Now we will summarize the studies that shed light on us in our thesis studies and what has been done in the studies.

In section 5.1, we solved a Pillai -type problem related to Lucas numbers[2]. The Pillai problem was formulated in 1936 [3,4].

Mihailescu solved this problem a special case of Pillai equation, called the Catalan conjecture [5]. Recently, Pillai-type problems involving recurrence number sequences have been investigated using elements of Baker's method [6,7,8,9,10,11].

In section 5.2, we solved the Brocard-Ramanujan equation associated with Mersenne numbers [12]. One of the famous Diophantine equations is the Brocard-Ramanujan equation. This problem is one of the still open problems of number theory. The only known solutions to are $(n, m) \in \{(4,5), (5,11), (7,71)\}$. This problem was proposed by Brocard and Ramanujan in *1876* and *1913* [13]. The solutions of this equation under some special conditions have been investigated by different mathematicians.

Gérardin put forward the idea that when $m > 71$, m must be a number at least *20* digits [14]. Lately, the Berndt and Galway proved that there are no other solutions up to $n = 109$ [15]. Furthermore, Marques studied the form of the equation for Fibonacci numbers [16]. Faco and Marques studied the case where the right-hand side of the equation is Tribonacci numbers [17]. Dabrowski and Ulas investigated some varieties of the Brocard-Ramanujan equation [18].

In section 5.3, we introduced Fermat-type equations. Fermat published this problem in his Arithmetica book in *1637*. But the solution to this problem is not provided in this book. Although mathematicians worked on this problem for many years, only the English mathematician Andrew Wiles succeeded in solving this problem after 300 years [19]. After that, many authors worked on different Fermat-type equations [20,21, 22].

Finally, we referenced some books that have been written about different types of Diophantine equations [23, 24].

PART 3

THEORETICAL BACKGROUND

3.1. Some Necessary Tools for Section 5.1.

First, we give a Baker-type lower bound for nonzero linear form in logarithms of algebraic numbers, which we will use three times to prove our result.

Definition 3.1.1. If a real (or complex) number m is the root of a polynomial, it is called an algebraic number.

Definition 3. 1. 2. Let m be an algebraic number. The logarithmic height of m is defined as

$$h(m) = d^{-1}\left(\sum_{i=1}^{d} \log(max\{1, |m_i|\}) + \log m_0\right)$$

where d is the degree of m and

$$g(X) = m_0 \prod_{i=1}^{d} (X - m_i) \in \mathbb{Z}[X]$$

is minimal primitive polynomial over the integers such that m is a root of this polynomial, and the leading coefficient m_0 is positive.

Theorem 3.1.3. [25] (Matveev) Let $\mathbb{L}$ be a real algebraic number field of degree D and $r_1, r_2, \ldots, r_k$ are positive real numbers of $\mathbb{L}$ and $u_1, u_2, \ldots, u_k$ are rational integers. Let's denote

$\Delta = \left(\prod_{i=1}^{k} r_i^{u_i}\right) - 1$
and supposing that $\Delta \neq 0$, then

$$|\Delta| > exp(-1{,}4 \times 30^{k+3} \times k^{4{,}5} \times D^2(1 + \log D)(1 + \log B)A_1 \dots A_k)$$

where
$B \geq max\{|u_1|, |u_2|, \dots, |u_k|\ \}$
and
$A_i \geq max\{Dh(r_i), |\log r_i|, 0.16\}, i = 1,2, \dots, k.$

Besides, we give the following lemma of Dujella and Petho, which will be the main tool we will use to reduce the upper bounds on the variables of our equation. Let Y be a real number. We denote the distance from Y to the nearest integer as follows

$$||Y|| = min\{|Y - n|, n \in \mathbb{Z}\}.$$

Lemma 3.1.4. [26] Let $M \in \mathbb{N}^*$ and let $\frac{p}{q}$ be a convergent of the continued fraction of the irrational γ such that $q > 6M$. Furthermore, suppose that C, K, μ are some real numbers with $C > 0$, $K > 1$ and let $\epsilon = ||\mu q|| - M||\gamma q||$. If $\epsilon > 0$, then there is no solution to the following inequality

$$0 < |s\gamma - t + \mu| < CK^{-r}$$

with positive integers s, t and r with

$$s \leq M \text{ and } r \geq \frac{\log \frac{Cq}{\epsilon}}{\log K}.$$

Let $(L_k)_{k \geq 1}$ be the Lucas sequence defined by the recurrence relation
$L_k = L_{k-1} + L_{k-2}$ for all $k \geq 3$, with initial conditions $L_1 = 1, L_2 = 3$.

Next, we will give some important properties of Lucas numbers.

Theorem 3.1.5. [27] (Binet's formula) Let $(L_k)_{k\geq 1}$ be the Lucas sequence, then

$$L_k = \alpha^k + \beta^k$$

where

$$(\alpha, \beta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right), and\ k \geq 1.$$

Lemma 3.1.6. [27] Let $k \geq 0$. Then

$L_{4k} \equiv 2\ (mod\ 5)$, $L_{4k+1} \equiv 1\ (mod\ 5)$,
and
$L_{4k+2} \equiv 3\ (mod\ 5)$, $L_{4k+3} \equiv 4\ (mod\ 5)$.

Lemma 3.1.7 Let t, u, l, k be some positive integers, then

1) $3^{2t-1} - 1 = 2A_t$
2) $3^{2t-1} + 1 = 4B_t$
3) $3^{2t} + 1 = 2C_t$
4) $3^{8t-2} - 1 = 8D_t$
5) $3^{8t-4} - 1 = 16F_t$
6) $3^{8t-6} - 1 = 8R_t$
7) $3^{8t} - 1 = 32S_t$, for all $t = 2k - 1$
8) $3^{8t} - 1 = 2^{l+4}K_t$, for all $t = u \cdot 2^{l-1}$.

Here, the odd numbers A_t, B_t, C_t, D_t, F_t, R_t, S_t, K_t and the integer Z_t are positive integers that change according to the value of t.

Proof.

1) $\mathbf{3^{2t-1} - 1 = (3-1)(3^{2t-2} + 3^{2t-3} + \cdots + 3 + 1) = 2A_t}$

3) $3^{2t} + 1 = 3 \cdot 3^{2t-1} + 1 = 3 \cdot (2A_t + 1) + 1 = 6A_t + 4 = 2(3A_t + 2) = 2C_t$

4) $$\begin{aligned} 3^{8t-4} - 1 &= (3^{2t-1} - 1)(3^{2t-1} + 1)(3^{4t-2} + 1) \\ &= (3^{2t-1} - 1)(3^{2t-1} + 1)\big((3^{2t-1} - 1)(3^{2t-1} + 1) + 2\big) \end{aligned}$$

$$= 8A_tB_t(8A_tB_t + 2) = 16F_t$$

8) $3^{2^{l+2}} - 1 = (3-1)(3+1)(3^2+1) \cdot \ldots \cdot \left(3^{2^{l+1}} + 1\right)$

If we use part 3), we get that

$$(3-1)(3+1)(3^2+1) \cdot \ldots \cdot \left(3^{2^{l+1}} + 1\right) = 2 \cdot 4 \cdot 2^{l+1}K_t = 2^{l+4}K_t.$$

and

$$3^{u \cdot 2^{l+2}} - 1 = \left(3^{2^{l+2}}\right)^u - 1 = 2^{l+4}Z_t.$$

We can prove the other parts of Lemma in the same way. From this Lemma we obtain following result:

If $3^t - 1 \equiv 0 \ (mod\ 2^l)$ and $l \geq 3$ then $t_{min} = 2^{l-2}$ and $t = u \cdot 2^{l-2}, u \geq 1$.

Using induction method, we can prove that the following lemma.

Lemma 3.1.8. If $l \geq 4$ and $u \geq 1$. Then:

$$3^{u \cdot 2^{l-2}} - 1 \equiv 0 \ (mod\ 5)$$

Lemma 3.1.9. [27] Let $(L_k)_{k \geq 1}$ be the Lucas sequence, then following inequality holds

$$\alpha^{k-2} \leq L_k \leq 2\alpha^{k-1}.$$

3.2. Some Necessary Tools for Section 5.2.

Definition 3.2.1. Mersenne numbers are defined by the recurrence relation and for all

$M_n = 2^n - 1, n \geq 1.$

Lemma 3.2.2. If $a, b > 0$ and $n \in \mathbb{N}^*$ then the below inequality is holds:

$$2^{n-1}(a^n + b^n) \geq (a + b)^n.$$

Lemma 3.2.3. If $n \in \mathbb{N}^*$ then:

$$\frac{2}{n+1} \leq (n!)^{\frac{-1}{n}} < \frac{e}{n+1} \tag{3.2}$$

holds.

Proof. Applying the Arithmetic-Geometric mean inequality [28] for the numbers $1, 2, \ldots, n$, we obtain

$$\frac{n+1}{2} \geq (n!)^{\frac{1}{n}},$$

or

$$\frac{2}{n+1} \leq (n!)^{\frac{-1}{n}}.$$

Now we must prove

$$(n!)^{\frac{-1}{n}} < \frac{e}{n+1}$$

We use the induction method for this. If $n = 1$ then we get $1 < \frac{e}{2}$. Suppose that inequality (3.2) is true for $n = k$ and show that it holds for $n = k + 1$. We know that

$$e^{\frac{1}{n+1}} \leq 1 + \frac{1}{n} \leq e^{\frac{1}{n}}, \tag{3.3}$$

then by our assumption and inequality (3.3), we have

$$\frac{1}{(k+1)!} < \left(\frac{e}{k+1}\right)^k \frac{1}{k+1} < \left(\frac{e}{k+2}\right)^{k+1}.$$

Lemma 3.2.4. If $n \in \mathbb{N}^*$ then:

$$\left(\frac{n+1}{3}\right)^n > 2^{2n} - 2^{n+1} \tag{3.4}$$

satisfies for all $n \geq 11$.

Proof. We suppose that $n = 11$. Then

$$4^{11} > 2^{22} - 2^{12},$$

that is true.
Besides, we assume that inequality (3.4) is satisfied for $n = k$. Then we must illustrate for $n = k + 1$

$$\left(\frac{k+2}{3}\right)^{k+1} > 2\left(\frac{k+1}{3}\right)^{k+1} = \frac{2(k+1)}{3}\left(\frac{k+1}{3}\right)^k > 2^{2k+3} - 2^{k+4} > 2^{2k+2} - 2^{k+2}.$$

Finally, we give a way to prime factorization of $n!$. To acquire the prime factorization of $n!$ we must find, for each of these primes p, the exponent g_p of the greatest power of p that divides $n!$. The method we will use is connected to the relation between n, p and g_p. Thanks to this method, the formula of g_p depending on n and p was determined, which also allows to find g_p for a given n and p in practise. This method belongs to the illustrious French mathematician A. Legendre [29].

To explain proposed method, let's start by giving the base p representation of the positive integer n. Assume this is given as

$$n = \sum_{i=0}^{k} r_i p^i,$$

where $p^{k+1} > n$, $p^k \leq n$ and $0 \leq r_i \leq p-1$, for all $i = 1,2,\dots,k$. Then, for l $(1 \leq l \leq k)$,

$$\frac{n}{p^l} = \frac{\sum_{i=l}^{k} r_i p^i}{p^l} + \frac{\sum_{i=0}^{l-1} r_i p^i}{p^l},$$

thus

$$\sum_{i=0}^{l-1} r_i p^i \leq (p-1)\sum_{i=0}^{l-1} p^i = p^l - 1 < p^l,$$

then, we get

$$\left[\frac{n}{p^l}\right] = \sum_{i=0}^{k-l} r_{k-i} p^{k-l-i}, \text{ for all } 1 \leq l \leq k. \tag{3.5}$$

In addition, we know that, given for any prime $p \leq n$,

$$g_p = \sum_{i=1}^{k}\left[\frac{n}{p^k}\right], \tag{3.6}$$

where $p^{k+1} > n$, $p^k \leq n$.

For each l $(1 \leq l \leq k)$, we write the formula (3.5) and add side-by-side and using the formula (3.6) we obtain

$$g_p = \sum_{i=1}^{k}\left(r_i \sum_{j=1}^{i-1} p^{j-1}\right) = \frac{1}{p-1}\sum_{i=1}^{k}\left(r_i\left(p^i - 1\right)\right) = \frac{n-d_p}{p-1}, \tag{3.7}$$

where $d_p = \sum_{i=0}^{k} r_i$.

3.3. Some Necessary Tools for Section 5.3.

Now we give the Fermat-Wiles theorem:

Theorem 3.3.1. [19] The below equation there is no solution in $\mathbb{N}^*$:

$$x^n = y^n + z^n$$

Definition 3.3.2. Fibonacci numbers are defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, and for all $n \geq 3$, with the initial conditions $F_1 = F_2 = 1$.

PART 4

METHODOLOGY

4.1. Factoring Method

Now let's explain the meaning of the Factoring method [24]. Let's look at the given equation

$$g(t_1, t_2, \dots, t_n) = 0. \qquad (4.1)$$

Now let's say that there are functions $g_1, g_2, \dots g_k \in Z[t_1,\ t_2, \dots, t_n]$, and the $s \in \mathbb{Z}$, so that equation (4.1) can be written in the following equivalent form by means of these parameters:

$$g_1(t_1, t_2, \dots, t_n) g_2(t_1, t_2, \dots, t_n) \cdots g_k(t_1, t_2, \dots, t_n) = s.$$

If the prime factorization of the number $s = s_1 s_2 \cdots s_k$ is given, then we get the following system that gives all the solutions of equation (4.1).

$$\begin{aligned} g_1 &= s_1 \\ g_2 &= s_2 \\ &\vdots \\ g_k &= s_k\ . \end{aligned}$$

Example 4.1.1. The below inequality there is no solution in positive integers:

$$a + a^2 + a^3 + a^4 + a^5 + a^6 = b^2$$

Proof. It is obvious that the above equation can be written in the following form

$a(1 + a^2 + a^3 + a^4 + a^5) = b^2,$

Since

$gcd(a, 1 + a^2 + a^3 + a^4 + a^5) = 1.$

Then, we obtain

$\exists\ x, y \in N^*$, $gcd(x, y) = 1$ and $b = xy$

$$\begin{cases} a = x^2 \\ 1 + a^2 + a^3 + a^4 + a^5 = y^2. \end{cases}$$

Therefore, we get that:

$a(1 + a^2 + a^3 + a^4) = y^2 - 1.$

Also, we see that:

$gcd(a, 1 + a^2 + a^3 + a^4) = 1.$

Hence, we should examine the following two cases:

i) $gcd(y - 1,\ y + 1) = 1.$

In this case we obtain that:

$$\begin{cases} a = y - 1 \\ 1 + a^2 + a^3 + a^4 = y + 1 \end{cases}$$

Hence, we get that:

$a^2 + a^3 + a^4 = a + 1.$

However, this equation there is no solution in the positive integers.

ii) $gcd(y-1,\ y+1) = 2$,

then

$\exists r_0, d_0, gcd(r_0, d_0) = 1,$

$y-1 = 2r_0,\ y+1 = 2d_0.$

Thus

$r_0 + 1 = d_0,$

hence

$a(1 + a^2 + a^3 + a^4) = 4r_0\, d_0.$

Then to find the solution to this equation, we must examine the following cases:

$a = 4, 1 + a^2 + a^3 + a^4 = r_0\, d_0 = r_0(r_0 + 1),$

there is no solution to the last equation. Because the left-hand side is odd but the right-hand side is an even number.

$a = r_0, 1 + a^2 + a^3 + a^4 = 4(r_0 + 1),$

then if we write r_0 instead of a in the second expression

${r_0}^4 + {r_0}^3 + {r_0}^2 = 4r_0 + 3.$

We can show that there is no solution to the last equation follows:
Let us define the following function on the set of positive integers:

$f(t) = t^4 + t^3 + t^2 - 4t - 3,$

clearly, this function is an increasing function and

$f(t) \geq 17, t \geq 2.$

Moreover, if

$$t = 1,\ f(t) = -4.$$

We illustrated that there is no solution in this case.
We now take

$a = 4r_0, 1 + a^2 + a^3 + a^4 = r_0 + 1.$

We clearly see that this is not possible. With this, it has been illustrated that the equation has no solutions in $\mathbb{N}^*$.

4.2. Inequalities method

The general purpose of this method is to solve the given Diophantine equation with an appropirate inequality [24]. The set of solutions of the given equation is narrowed by the inequality appropriate to the equation and the solutions are found.
Example 4. 2. 1. Find all solutions of below equation in $\mathbb{N}^*$.

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

Solution. Obviously, $x, y, z > 1$. Without compromising the generality, we may assume that $x \geq\ y \geq\ z$. The implies $z \leq 3$.

If $z = 2$, then $\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$ with $y \leq 4$. Then we get the solutions $\{(6,3,2),(4,4,2)\}$.

If $z = 3$, then $\frac{1}{x} + \frac{1}{y} = \frac{2}{3}$ with $y \leq 3$, and the solutions $\{(6,2,3),(3,3,3)\}$.

PART 5

MAIN RESULT

5.1. A Pillai-type Problem Associated with Lucas Numbers

This part of our dissertation was published under the title A Pillai-type Problem Associated with Lucas Numbers in Pak-Turk conference 2023[2].

Theorem 5.1.1. The only positive integer solutions of the following equation are $(x, y, n) = \{(1,1,1), (2,3,1), (2,1,3)\}$

$$3^x - L_n 2^y = 1 \tag{5.1}$$

Proof. From Lemmas 3.1.6, 3.1.7, and 3.1.8 we get that there are no solutions to equation (5.1) when $y \geq 4$. Therefore we have to investigate the cases $y = 1$, $y = 2$, and $y = 3$.

$i)$ $y = 1$.
By Lemma 3.1.9, we get

$$6\alpha^{n-1} \geq 1 + 2L_n = 3^x$$

or

$$x < \frac{\log 6}{\log 3} + (n-1)\frac{\log \alpha}{\log 3} < 1{,}64 + 0{,}44(n-1)\ .$$

From last inequality, we conclude that, when $n \geq 3$, if the equation has a solution then $x < n$. When $n = 1,2,3$ we obtain the trivial $(x, y, n) = \{(1,1,1), (2,1,3)\}$ solutions of (5.1). By Binet's formula

$$3^x - 2L_n = 3^x - 2(\alpha^n + \beta^n) = 1,$$

then

$$|3^x - 2\alpha^n| = |1 + 2\beta^n| \leq 1 + 2|\beta|^n < 4$$

which implies

$$|3^x \alpha^{-n} 2^{-1} - 1| < \frac{2}{\alpha^n}. \qquad (5.2)$$

We put

$$\Delta_1 = 3^x \alpha^{-n} 2^{-1} - 1,$$

It is obvious that, $\Delta_1 \neq 0$, because if $\Delta_1 = 0$, then $\alpha^n \in \mathbb{Q}$ which is false. To find the lower bound for Δ_1 we use Matveev's theorem, we take

$$k = 3, r_1 = 3, r_2 = \alpha, r_3 = 2, u_1 = x, u_2 = -n, u_3 = -1.$$

r_1, r_2, r_3 are algebraic numbers of $L = \mathbb{Q}(\sqrt{5})$ which is degree 2. According to the definition of the logarithmic height of algebraic numbers, we get

$$h(r_1) = \log 3, h(r_2) = \frac{1}{2}\log\alpha\,, h(r_3) = \log 2,$$

then we can take

$$A_1 = 2\log 3,\ A_2 = \log\alpha\,, A_3 = 2\log 2.$$

Finally, we know that $x < n$ then we can take $B = n$. Then, by Theorem 3.1.3, we have that

$$\log|\Delta_1| > -1{,}4 \cdot 30^6 \cdot 3^{4{,}5} \cdot 4 \cdot (1 + \log 2)(1 + \log n)(2\log 3\,)(\log\alpha)(2\log 2)$$

by comparing last inequality with (5.2), we obtain

$$n \log \alpha - \log 2 < 1{,}4 \cdot 30^6 \cdot 3^{4{,}5} \cdot 4(1 + \log 2)(1 + \log n)(2 \log 3\,)(\log \alpha)(2 \log 2),$$

Then

$$n < 1{,}5 \cdot 30^6 \cdot 3^{4{,}5} \cdot 4 \cdot (1 + \log 2)(1 + \log n)(2 \log 3\,)(2 \log 2).$$

Hence, we get $n < 1{,}06 \cdot 10^{14}$.

Next, we apply the result of Dujella and Petho to reduce the above bound for n. From inequality (5.2), we conclude that $|\Delta_1| < \frac{2}{\alpha^n} < \frac{1}{2}$ for all $n \geq 3$ and we know that $2|z| > |\log(1 + z)|$ holds for all $z \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. Hence, we get

$$\frac{|x \log 3 - n \log \alpha - \log 2|}{2} < \frac{2}{\alpha^n}$$

Therefore, we obtain

$$\left| x \frac{\log 3}{\log \alpha} - n - \frac{\log 2}{\log \alpha} \right| < \frac{9}{\alpha^n}$$

We now apply Lemma 3.1.4 with the following data

$$\gamma = \frac{\log 3}{\log \alpha}, s = x, t = n, \mu = -\frac{\log 2}{\log \alpha}, C = 9, K = \alpha, r = n.$$

In addition, since $x < n < 1{,}06 \cdot 10^{14}$ we can take $M = 1{,}06 \cdot 10^{14}$. The $33th$ convergent to γ

$$\frac{p_{33}}{q_{33}} = \frac{4509703705422533}{1975330854159075}$$

Then, $q = q_{33} = 1975330854159075 > 6M$. Hence, it gives $\epsilon > 0{,}202$.

Then if the equation has solutions, then

$$n < \frac{\log(9 \cdot 1975330854159075)/0{,}202)}{\log \alpha} < 82.$$

A ran using Mathematica showed that equation (5.1) has no solution under the conditions $y = 1$, $x < n < 82, n > 3$.

ii) $y = 2$.

By Lemma 3.1.9, we get

$$10\alpha^{n-1} \geq 1 + 4L_n = 3^x$$

or

$$x < \frac{\log 10}{\log 3} + (n-1)\frac{\log \alpha}{\log 3} < 2{,}1 + 0{,}44(n-1),$$

Therefore, when $n \geq 3$, if the equation has a solution then $x < n$. It is clear that when $n = 1,2,3$ there are no solutions to equation (5.1). By Binet's formula

$$3^x - 4L_n = 3^x - 4(\alpha^n + \beta^n) = 1,$$

then

$$|3^x - 4\alpha^n| = |1 + 4\beta^n| \leq 1 + 4|\beta|^n < 6,$$

which implies

$$|3^x\alpha^{-n}4^{-1} - 1| < \frac{2}{\alpha^n}. \tag{5.3}$$

We put

$$\Delta_2 = 3^x\alpha^{-n}4^{-1} - 1,$$

obviously, $\Delta_2 \neq 0$, because if $\Delta_2 = 0$, then $\alpha^n \in \mathbb{Q}$, which is contradiction. We now apply Matveev's result to get a lower bound for $|\Delta_2|$, we take

$$k = 3, r_1 = 3, r_2 = \alpha, r_3 = 4, u_1 = x, u_2 = -n, u_3 = -1.$$

r_1, r_2, r_3 are algebraic numbers of $L = \mathbb{Q}(\sqrt{5})$ which is degree $D = 2$. By the definition of the logarithmic height of algebraic numbers, we get

$$h(r_1) = \log 3, h(r_2) = \frac{1}{2}\log \alpha\,, h(r_3) = \log 4,$$

then we can take

$$A_1 = 2\log 3,\ A_2 = \log \alpha\,, A_3 = 2\log 4.$$

Additonally, we know that $x < n$ then we can take $B = n$. Then, by Theorem 3.1.3, we obatin

$$\log|\Delta_1| > -1{,}4 \cdot 30^6 \cdot 3^{4{,}5} \cdot 4 \cdot (1 + \log 2)(1 + \log n)(2\log 3\,)(\log \alpha)(2\log 4),$$

by comparing this with inequality (5.3), we acquire

$$n\log \alpha - \log 2 < 1{,}4 \cdot 30^6 \cdot 3^{4{,}5} \cdot 4(1 + \log 2)(1 + \log n)(2\log 3\,)(\log \alpha)(2\log 4),$$

thus

$$n < 1{,}5 \cdot 30^6 \cdot 3^{4{,}5} \cdot 4 \cdot (1 + \log 2)(1 + \log n)(2\log 3\,)(2\log 4),$$

hence, we get $n < 2.2 \cdot 10^{14}$.

We again use the result of Dujella and Petho. By inequality (5.3) we have that $|\Delta_2| < \frac{2}{\alpha^n} < \frac{1}{2}$ for all $n \geq 3$ and we know that $2|z| > |\log(1+z)|$ satisfies for all $z \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. Thus, we conclude that

$$\frac{|x\log 3 - n\log\alpha - \log 4|}{2} < \frac{2}{\alpha^n},$$

then, by dividing both sides of last inequality by $\log\alpha$, we obtain that

$$\left|x\frac{\log 3}{\log\alpha} - n - \frac{\log 4}{\log\alpha}\right| < \frac{9}{\alpha^n}.$$

We now apply Lemma 3.1.4 with the following data

$$\gamma = \frac{\log 3}{\log\alpha}, s = x, t = n, \mu = -\frac{\log 4}{\log\alpha}, C = 9, K = \alpha, r = n.$$

In addition, since $x < n < 2{,}2 \cdot 10^{14}$ we can choose $M = 2{,}2 \cdot 10^{14}$. We know, the $33th$ convergent to γ

$$\frac{p_{33}}{q_{33}} = \frac{4509703705422533}{1975330854159075}$$

then, $q = q_{33} = 1975330854159075 > 6M$. According to these data, we get that $\epsilon > 0{,}411$. Then if the equation has solutions, then

$$n < \frac{\log(9 \cdot 1975330854159075)/0{,}411)}{\log\alpha} < 80.$$

A run with the Mathematica program showed there are no solutions to equation (5.1) in the range $x < n < 80, n > 3$. We completed the analysis in the case $y = 2$.

iii) $y = 3$.

From Lemma 3.1.9, we have that

$$18\alpha^{n-1} \geq 1 + 8L_n = 3^x$$

or

$$x < \frac{\log 18}{\log 3} + (n-1)\frac{\log \alpha}{\log 3} < 2{,}64 + 0{,}44(n-1)$$

Thus, when $n \geq 4$, if the equation has a solution then $x < n$. For cases $n = 1, 2, 3$ we have the trivial $(x, y, n) = (2, 3, 1)$ solution of equation (5.1). Using Binet's formula, we can write equation (5.1) as follows

$$3^x - 8L_n = 3^x - 8(\alpha^n + \beta^n) = 1,$$

then

$$|3^x - 8\alpha^n| = |1 + 8\beta^n| \leq 1 + 8|\beta|^n < 10.$$

Hence, we obtain

$$|3^x \alpha^{-n} 8^{-1} - 1| < \tfrac{5}{4\alpha^n}\,. \tag{5.4}$$

We put,

$$\Delta_3 = 3^x \alpha^{-n} 8^{-1} - 1,$$

if $\Delta_3 = 0$, then $\alpha^n \in \mathbb{Q}$, which is not true. This means that $\Delta_3 \neq 0$. We apply Matveev's theorem one more time to find the lower bound for Δ_3, we take

$$k = 3, r_1 = 3, r_2 = \alpha, r_3 = 8, u_1 = x, u_2 = -n, u_3 = -1.$$

Clearly, $r_1, r_2, r_3 \in L = \mathbb{Q}(\sqrt{5})$. Then, we take $D = 2$. From the definition of the logarithmic height of algebraic numbers, we obtain that

$$h(r_1) = \log 3, h(r_2) = \tfrac{1}{2}\log \alpha\,, h(r_3) = \log 8\,,$$

then we can choose

$A_1 = 2\log 3,\ A_2 = \log\alpha\,, A_3 = 2\log 8.$

Finally, we know that $x < n$ then we can take $B = n$. Then, by Theorem 3.1.3, we conclude that

$$\log|\Delta_3| > -1{,}4\cdot 30^6\cdot 3^{4{,}5}\cdot 4\cdot(1+\log 2)(1+\log n)(2\log 3\,)(\log\alpha)(2\log 8),$$

by comparing this with inequality (5.4), we obtain that

$$n\log\alpha - \log 2 < 1{,}4\cdot 30^6\cdot 3^{4{,}5}\cdot 4(1+\log 2)(1+\log n)(2\log 3\,)(\log\alpha)(2\log 8),$$

then

$$n < 1{,}5\cdot 30^6\cdot 3^{4{,}5}\cdot 4\cdot(1+\log 2)(1+\log n)(2\log 3\,)(2\log 8).$$

Hence, we obtain $n < 3{,}27\cdot 10^{14}$.

Next, We use the result of Dujella and Petho once again to reduce the upper bound of n. We get from inequality (5.4)

$|\Delta_3| < \frac{1{,}25}{\alpha^n} < \frac{1}{2}$ for all $n \geq 3$.

We have

$2|z| > |\log(1+z)|$ holds for all $z \in \left(-\frac{1}{2},\frac{1}{2}\right)$.

So, we get

$$\frac{|x\log 3 - n\log\alpha - \log 8|}{2} < \frac{1.25}{\alpha^n}$$

then, we obtain

$\left|x\frac{\log 3}{\log \alpha} - n - \frac{\log 8}{\log \alpha}\right| < \frac{6}{\alpha^n},$

we again apply Lemma 3.1.4 with the following data

$\gamma = \frac{\log 3}{\log \alpha}, s = x, t = n, \mu = -\frac{\log 8}{\log \alpha}, C = 6, K = \alpha, r = n.$

Therefore, since $x < n < 3{,}27 \cdot 10^{14}$ we can take $M = 3{,}27 \cdot 10^{14}$. The $33th$ convergent to γ

$\frac{p_{33}}{q_{33}} = \frac{4509703705422533}{1975330854159075},$

obviously, $q = q_{33} = 1975330854159075 > 6M$. Hence, we get $\epsilon > 0{,}25$. Then if the equation has solutions, then

$n < \frac{\log(9 \cdot 1975330854159075)/0{,}25)}{\log \alpha} < 80.$

A search using Mathematica showed there are no solutions to equation (5.1) when $x < n < 80$ and $n > 3$. We have completed the proof Theorem 5.1.1.

5.2. Mersenne Version of Brocard-Ramanujan Equation

This part of our thesis was published under the title Mersenne version of Brocard-Ramanujan equation in Journal of New Results in Science[12].

Theorem 5.2.1. The following eqaution there is no solution in $\mathbb{N}^*$:

$$n! + 1 = {M_k}^2 \tag{5.5}$$

Proof. We investigate following cases:

i) We suppose that $n = k$. Then we must demonstrate there is no solution to the following equation

$$n! = 2^{2n} - 2^{n+1}. \tag{5.6}$$

We can demonstrate by simple numerical calculations that there is no solution of equation (5.6) for all $n \leq 10$. Now we consider $n \geq 11$.

From (3.4) we have

$$2^{2n} - 2^{n+1} < \left(\frac{n+1}{3}\right)^n,$$

on the other side from (3.2) we have

$$n! > \left(\frac{n+1}{3}\right)^n.$$

From last two inequalities, we get that Equation (5.6) has no solution in the case of $n \geq 11$.

ii) We assume that $n > k$. Then we must prove there is no solution to the following equation

$$n! = 2^{2k} - 2^{k+1}, \tag{5.7}$$

According to (3.4) we have following inequality

$$n! > 2^{2n} - 2^{n+1}$$

satisfies for all $n \geq 11$.

In addition, we know $n > k$. Then we get

$n! > 2^{2n} - 2^{n+1} > 2^{2k} - 2^{k+1}$,

for all (k,n) pairs satisfying $11 \leq k < n$ or $k < 11 \leq n$. Then, from last inequality we obtain there is no solution to the equation (5.7) for both cases of n and k. Besides, for the remaining case $k < n < 11$, it can be shown by simple mathematical calculations that the equation has no solution.

iii) Suppose that $n < k$. We see that the right-hand side of the equation is divided by 2^{k+1} and from the formula (3.7), we obtain that the left-hand side is divided by 2^{n-d_2}. However, we see that $n - d_2 < n < k < k + 1$. This means that, in this case, the equation has no solution.

5.3. Some New Fermat Type Equation

Theorem 5.3.1. If $k, n \in \mathrm{N}^*$, $n \geq 3k$ and $k|n$ then the following equation there is no solution in N^*:

$$a^n = (bc)^k(b^{n-2k} + c^{n-2k}). \tag{5.8}$$

Proof. Assume that $gcd(b,c) = d$ then $\exists p, q \in \mathrm{N}^*$ such that $b = pd$ and $c = qd$, $gcd(p,q) = 1$. Then we get

$$a^n = (pq)^k d^n (p^{n-2k} + q^{n-2k}),$$

hence $d^n | a^n$ then $d|a$. Let $m = \frac{a}{d} \in N^*$ then we acquire

$$m^n = (pq)^k (p^{n-2k} + q^{n-2k}).$$

Since $gcd(p,q) = 1$ we get

$$gcd(p,q) = gcd(p^k,\ p^{n-2k} + q^{n-2k}) = gcd(q^k,\ p^{n-2k} + q^{n-2k}) = 1,$$

therefore $\exists r, s, z \in \mathrm{N}^*$ such that:

$p^k = r^n, q^k = s^n, p^{n-2k} + q^{n-2k} = z^n,$

then we obtain

$$p^{n-2k} + q^{n-2k} = z^n = \left(r^{\frac{n-2k}{k}}\right)^n + \left(q^{\frac{n-2k}{k}}\right)^n.$$

But according to Theorem 3.3.1, the last equation has no solution in the set of positive integers. So, we proved that the equation (5.8) there is no solution in positive integers.

In the case of $k = 1$ in Theorem 5.3.1, we obtain the following interesting result:

Result 5.3.2. If $n \in \mathrm{N}^*$ and $n \geq 3$ the below equation there is no solution in N^*:

$$a^n = bc(b^{n-2} + c^{n-2}). \tag{5.9}$$

Besides, inspired by Theorem 3.3.1 and Result 5.3.2, we propose an interesting conjecture:

Conjecture 5.3.3. If $n \in \mathrm{N}^*$ and $n \geq 5$ then the following equation there is no solution in positive integers:

$$2a^n = (b + c)(b^{n-1} + c^{n-1}).$$

Furthermore, prove that the conjecture holds in some special cases.

Theorem 5.3.4. The below equation there is no solution in positive integers:

$$2a^4 = (b + c)(b^3 + c^3)$$

Proof. It is obvious that

$$2a^4 = (b + c)(b^3 + c^3) = (b + c)^2(b^2 - bc + c^2),$$

we suppose that $gcd(b,c) = d$. Then $\exists u, v \in \mathrm{N}^*$ such that $b = ud$ and $c = vd$, $gcd(u,v) = 1$. Hence

$$2a^4 = d^4(u+v)^2(u^2 - uv + v^2),$$

then $\exists m \in \mathrm{N}^*$, $m = \frac{a}{d}$.

Hence

$$2m^4 = (u+v)^2(u^2 - uv + v^2). \qquad (5.10)$$

In addition, assuming that

$$\exists t \in \mathrm{N}^*, gcd((u+v)^2,\ u^2 - uv + v^2) = t,$$

knows $gcd(u,v) = 1$,

$$u^2 - uv + v^2 = (u+v)^2 - 3uv.$$

Then we obtain $t|3$ that means either $t = 1$ or $t = 3$.
Then let's investigate the following cases:

i) If we take, $t = 1$ then we obtain $\exists p, q, r \in \mathrm{N}^*$

$$u^2 - uv + v^2 = 2^{4r-3}p^4, u + v = q^2,$$

from (5.10), we get

$$(q^2 - v)^2 - (q^2 - v)v + v^2 = 2^{4r-3}p^4$$

or

$$q^4 - 3q^2v + 3v^2 = 2^{4r-3}p^4.$$

There is no solution of last equation because the right side of the equation is even number, but the left side is odd number. Therefore, equation (5.10) has no solution for this case.

ii) If we take, $t = 3$ then we get $\exists x, y \in \mathbb{N}^*$ with $gcd(x, y) = 1$.

$$u^2 - uv + v^2 = 3y, u + v = 3x, \ 3 \nmid y. \tag{5.11}$$

According to m^4 is on the left-hand side of the equation (5.10), it must be

$$3^{4r} | (u+v)^2(u^2 - uv + v^2), r \in \mathbb{N}^*,$$

in addition, thanks to (5.11) we obtain

$$3^{2z+1} | (u+v)^2(u^2 - uv + v^2), z \in \mathbb{N}^*.$$

Hence, we have shown that equation (5.10) has no solution for this case.

Theorem 5.3.5. If $n \geq 5$ then the following equation there is no solution in $\mathbb{N}^*$:

$$2F_m^n = F_{l+2}(F_{l+1}^{n-1} + F_l^{n-1}) \tag{5.12}$$

Proof. We have
$(a+b)^n > a^n + b^n$,

is holds for $\forall a, b > 0$ and $n \in \mathbb{N}^*$. Then

$$F_{l+2}^{n-1} > F_{l+1}^{n-1} + F_l^{n-1},$$

thus

$$2F_m^n < F_{l+2}^n$$

or

$$\left(\frac{F_{l+2}}{F_m}\right)^n > 2.$$

Thanks to the last inequality we get

$$m < l + 2. \tag{5.13}$$

We now apply Lemma 3.2.2. for the right-hand side of our equation

$$2^{n-2}(F_l^{n-1} + F_{l+1}^{n-1}) \geq F_{l+2}^{n-1},$$

thus

$$2^{n-1}F_m^n \geq F_{l+2}^n,$$

then

$$\frac{F_{l+2}}{F_m} \leq 2^{\frac{n-1}{n}} < 2,$$

but

$$\frac{F_{l+2}}{F_l} > 2, \forall l \in \mathbb{N}^*.$$

Then we obtain

$$m \geq l + 1. \tag{5.14}$$

From equations (5.13) and (5.14) we get $m = l + 1$.

Then we have to investigate the solutions to the following equation

$$2F_{l+1}^{n} = F_{l+2}(F_{l+1}^{n-1} + F_{l}^{n-1})$$

or

$$F_{l+1}^{n-1}F_{l-1} = F_{l}^{n-1}F_{l+2} \text{ ,}$$

based on the divisibility properties of Fibonacci numbers, we get

$$gcd(F_{l+1}^{n-1}, F_{l+2}) = gcd(F_{l}^{n-1}, F_{l-1}) = 1.$$

Then we obtain that the last equation there is no solution. This means that, equation (5.12) has no solution.

PART 6

SUMMARY

In this study, we introduced some interesting exponential Diophantine equations. Then we studied integer solutions of these equations using different solving methods of Diophantine equations.

Firstly, we solved a Pillai-type equation associated with Lucas numbers using elements of Baker's theory and properties of Lucas numbers.

Secondly, we solved the Brocard-Ramanujan equation associated with Mersenne numbers using prime factorization of $n!$ and inequalities.

Finally, we introduced some new Fermat-type equations associated with Fibonacci numbers and investigated their integer solutions using inequalities and factoring method.

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RESUME

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