SOME OPTIMALITY CONDITIONS FOR INTERVAL-VALUED OPTIMIZATION PROBLEMS USING SUBDIFFERENTIALS

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Fouad Qasim AHMED

ABSTRACT

M. Sc. Thesis

SOME OPTIMALITY CONDITIONS FOR INTERVAL-VALUED OPTIMIZATION PROBLEMS USING SUBDIFFERENTIALS

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In this thesis, interval-valued numbers, interval-valued functions, the importance of interval-valued optimization problems, some applications of interval-valued optimization and solutions of interval-valued optimization and their solutions with subdifferentials are discussed. In the second part, some features and definitions of intervals, notation used for intervals, algebraic operations defined on intervals, interval-valued functions, some properties of interval-valued functions, metric and norm definitions defined on intervals, limit, continuity, derivative and integral of interval-valued functions are recalled and examined on examples. Interval-valued optimization, which is the finding of the extreme points of the interval-valued function on a domain, and the types of solutions and how to find these solutions are given in the third chapter. In addition, how find these solutions is examined on the examples.

Necessary or sufficient optimality conditions for optimization problems give some information about the solutions to the problem, or they give some conditions such that candidate points satisfy these to be solutions. In the last chapter, some optimality conditions for interval-valued optimization problems are obtained using subdifferentials.

Key Words : Interval-valed numbers, interval-valued functions, interval-valued optimization, solution methods, subdifferential
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ÖZET

Yüksek Lisans Tezi

ARALIK DEĞERLİ OPTİMİZASYON PROBLEMLERİ İÇİN SUBDİFERANSİYEL İLE BAZI OPTİMALLİK KOŞULLARI

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Bu tezde, aralık değerli sayılar, aralık değerli fonksiyonlar, aralık değerli optimizasyon problemlerinin önemi, aralık değerli optimizasyonun bazı uygulamaları ve aralık değerli optimizasyonun çözümleri ve subdiferansiyeller ile çözümleri ele alınmıştır. İlk bölümde aralık değerli sayıların öneminden bahsedildi. İkinci bölümde, aralıkların bazı özellikleri ve tanımları, aralıklar için kullanılan notasyon, aralıklar üzerinde tanımlanan cebirsel işlemler, aralık değerli fonksiyonlar, aralık değerli fonksiyonların bazı özellikleri, aralıklar üzerinde tanımlanan metrik ve norm tanımları, limit, süreklilik, türev ve aralık değerli fonksiyonların integralleri hatırlanmakta ve örnekler üzerinde incelenmektedir. Aralık değerli fonksiyonun bir tanım kümesi üzerindeki ekstremum noklarının bulunması olan aralık değerli optimizasyon, çözüm türleri ve bu çözümlerinin nasıl bulunacağı üçüncü bölümde verilmiştir. Ayrıca, bu çözümlerin nasıl bulunacağı örnekler üzerinde incelenmiştir.

Optimizasyon problemleri için gerekli veya yeterli optimallik koşulları, problemin çözümleri hakkında bazı bilgiler verir veya aday noktaların bunları çözüm olarak karşılaması için bazı koşullar verir. Son bölümde, aralık değerli optimizasyon problemleri için bazı optimallik koşulları subdiferansiyeller kullanılarak elde edilmiştir.

Anahtar Kelimeler **:** Aralık değerli sayılar, aralık değerli fonksiyon, aralık değerli optimizasyon, çözüm yöntemleri, subdiferansiyel

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SYMBOLS

$\mathbb{R}$: real numbers

$\mathbb{N}$: natural numbers

$I(\mathbb{R})$: the set of all closed intervals on real numbers

$-_g$: the generalized Hukuhara difference

F : an interval function

A^+ : the positive part of a closed interval A

A^- : the negative part of a closed interval A

$F'(x_0)$: GH-derivative of an interval function F at x_0

$F'_H(x_0)$: gH-derivative of an interval function F at x_0

$F'_h(x_0)$: gH-directional derivative of F at x_0 in the direction h

$\partial^s F(x_0)$: the s-subdifferential of interval function F at x_0

$\partial^s_w F(x_0)$: the weak s-subdifferential of interval function F at

$\partial^{mr} F(x_0)$: the mr-subdifferential of interval function F at x_0

$\partial^{mr}_w F(x_0)$: the weak mr-subdifferential of interval function F at x_0

$\mathbb{B}(a, b)$: the closed ball with radius b centered at a

$int(X)$: interior of set X

ABBREVIATIONS INDEX

(IVP) : Interval-valued Optimization Problem

Iff : If and only if

Wrt : With respect to

PART 1

INTRODUCTION

Optimization is a tool that can be used in a variety of businesses and functional domains. Optimization problems are important because they are used in a wide area of study domains. For example, we want to buy a phone. Then, we have a lot options. Which phone is suitable for us? Finding the best among the options is an optimization problem. So, we can see the optimization in statistic, economy, engineering, etc. Also, optimization is a powerful subject of mathematics.

We have distinct types of problems depending on the coefficients of the objective function: Deterministic problems when they are integers, stochastic problems when they are random parameters with known distributions, set-valued optimization when they are sets, and interval problems when they are closed intervals. The last type of problem is just as important as the others. Because the intervals are special sets, we can say that interval optimizations are a special part of set optimization. Similarly, because the intervals are a generalization of integers, we can say that deterministic optimization is a special form of the interval optimization. Recently, solving problems that occur under certain uncertainties has received more attention. These uncertainties can be thought of as uncertain weather conditions, traffic. The function that we search the best points under uncertainty is an interval-valued function. The problem of finding the best points of interval-valued functions is called interval-valued optimization. Interval-valued optimization problems have many applications in daily life. For example, Abbasi Molai and Khorram (2007) characterized the amount of 2 types of feed that should be given to feed the chickens with the least cost in a farm with 1000 chickens, with interval value optimization. Karaman (2021b) examined the problem of an investor with 500 000 TL, who wants to store his money in foreign currencies (Dollar, Euro, Sterling) to protect the value of his money, with interval value optimization. This problem is one of the most common problems in daily life. Karaman (2021b) found the solution of this problem by characterizing with interval-value optimization. Also, we can see a lot of applications of interval-valued optimization in the literature and there in references.

PART 2

INTERVALS

Some fundamental definitions, notations and propositions will be given and explained with examples in this chapter. Therefore, the necessary information will be collected for interval-valued optimizations.

2.1. INTERVALS AND ARICTHMETICS

During this, a closed and bounded interval A on real numbers ($\mathbb{R}$) is defined as:

$$A = [a_1, a_2] := \{x \in \mathbb{R} \mid a_1 \leq x \leq a_2\}$$

where the end points of the interval A is a_1 and a_2 (left and right) respectively. Also, interval A can write using closed ball as:

$$A = [a_1, a_2] = \mathbb{B}\left(\frac{a_1 + a_2}{2}, \frac{a_2 - a_1}{2}\right) = \left\{x \in \mathbb{R} : \left|x - \frac{a_1 + a_2}{2}\right| \leq \frac{a_2 - a_1}{2}\right\}$$

where $\mathbb{B}\left(\frac{a_1+a_2}{2}, \frac{a_2-a_1}{2}\right)$ is closed ball with radius $\frac{a_2-a_1}{2}$ centered at $\frac{a_1+a_2}{2}$.

The interval as can be used to define each real integer as $a = [a, a]$ for all $a \in \mathbb{R}$. An interval can be characterized as parametrically:

$$A = [a_1, a_2] = \{k \in \mathbb{R} \mid k = a_1 + t(a_2 - a_1), t \in [0,1]\}.$$

Now, let's recall some set properties of intervals. Let intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$ be given. A equals to B, that is $A = B$ if and only if (shortly, iff) $a_1 = b_1$ and $a_2 = b_2$.

If $a_1 \leq 0 \leq a_2$, then the negative and the positive parts of A are defined as: $A^- :=$ $[a_1, 0]$ and $A^+ := [0, a_2]$, respectively. Then $A = A^+ \cup A^-$.

The following properties are satisfied [12]:

- $a_1 > b_2$ or $b_1 > a_2 \Leftrightarrow A \cap B = \emptyset$
- $A \cup B = [\min\{a_1, b_1\}, \max\{a_2, b_2\}]$
- $A \subseteq B \Leftrightarrow b_1 \leq a_1$ and $a_2 \leq b_2$

If $a_1 = a_2$ then the interval A, named degenerate, so it equals to a single point a_1 or a_2. If $a_1 = -a_2$ for any interval $A = [a_1, a_2]$, then interval A is called symmetric interval. For example, $A = [-3,3]$ is a symmetric interval. The set of all closed and bounded intervals in $\mathbb{R}$ is indicated in this work by the character $I(\mathbb{R})$. Now, we recall some interval operations used in the interval analysis.

Let $A = [a_1, a_2], B = [b_1, b_2] \in I(\mathbb{R})\ and\ k \in \mathbb{R}$.

The sum or addition of intervals A and B is defined by:

$$A + B := \{x + y \in \mathbb{R} \mid x \in A\ and\ y \in B\} = [a_1 + b_1, a_2 + b_2],$$

the difference of two intervals A and B is given as:

$$A - B := \{x - y \in \mathbb{R} \mid x \in A\ and\ y \in B\} = [a_1 - b_2, a_2 - b_1],$$

and the scalar multiplication is defined by:

$$kA := \begin{cases} [ka_1, ka_2]\ ;\ k \geq 0 \\ [ka_2, ka_1];\ k < 0 \end{cases}$$

Multiplication of intervals A and B is also an interval, and defined as:

$$AB := [\min\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}, \max\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}].$$

Let's consider any interval $A = [a_1, a_2] \in I(\mathbb{R})$. The absorbing element for multiplication in $I(\mathbb{R})$ is $[0,0]$ because $A\ 0 = [a_1, a_2][0,0] = [0,0]$ satisfies. Moreover, $[0,0]$ is identity for addition in $I(\mathbb{R})$. Identity element for multiplication in $I(\mathbb{R})$ is $[1,1]$.

We need the inverse of multiplication for intervals to define the division of two intervals. Let us consider interval A. The reciprocal or multiplicative inverse is defined as:

$$\frac{1}{A} := \left\{\frac{1}{a} : a \in A\right\}.$$

Because $\frac{1}{0}$ is undefined, 0 is not belong to interval A. Then, we can only find reciprocal of intervals that do not contain 0. Then, we can obtain the following rules: For $A = [a_1, a_2\,] \in I(\mathbb{R})$,

$$\text{if } a_1 > 0 \text{ or } a_2 < 0, \text{ then } \frac{1}{A} = \left[\frac{1}{a_2}, \frac{1}{a_1}\right].$$

Division of intervals defined as: For $A = [a_1, a_2\,]$, $B = [b_1, b_2\,] \in I(\mathbb{R})$ and $0 \notin B$

$$\frac{A}{B} = A\frac{1}{B} = \left[\min\left\{a_1\frac{1}{b_1}, a_1\frac{1}{b_2}, a_2\frac{1}{b_1}, a_2\frac{1}{b_2}\right\}, \max\left\{a_1\frac{1}{b_1}, a_1\frac{1}{b_2}, a_2\frac{1}{b_1}, a_2\frac{1}{b_2}\right\}\right].$$

The intervals satisfy the cancellation law for interval addition, that is, $A + C = B + C$ implies $A = B$ for all $A, B, C \in I(\mathbb{R})$ [12]. But, the cancellation law for multiplication on intervals may not be satisfied. That is, $AC = BC \nRightarrow A = C$ for some $A, B, C \in I(\mathbb{R})$.

Now, let us give a power of an interval as: For interval $A = [a_1, a_2\,]$ and $t \in \mathbb{R}$ [4]:

- $A^0 = [1,1]$
- $A^t = A^{t-1}A$ if $t > 0$
- $A^{-t} = (A^{-1})^t$ if $0 \notin A$ and $t \geq 0$

Note that $A^n = \{a^n : a \in A\}$ may not be satisfied for all intervals A. For example,

$$[-3,3]^2 = [-3,3][-3,3] = [-9,9] \neq \{a^2 : a \in [-3,3]\} = [0,9].$$

Proposition 2.1. Let A be an interval and $t \in (0,1)$. Then, $tA + (1-t)A = A$

Proof: Let $A = [a_1, a_2]$. Then, $tA + (1-t)A = t[a_1, a_2] + (1-t)[a_1, a_2] = [ta_1 + (1-t)a_1, ta_2 + (1-t)a_2] = [a_1, a_2] = A$.

Now we give some notations used in the interval analysis.

Let $A = [a_1, a_2] \in I(\mathbb{R})$. Then, the radius, the midpoint or center and width of the interval A are defined by

$$r(A) := \frac{1}{2}(a_2 - a_1),$$

$$m(A) := \frac{1}{2}(a_1 + a_2)$$

and

$$w(A) := a_2 - a_1,$$

respectively. Then, the radius and the midpoint of the interval A are show by A_r and A_m, respectively. An interval can be expressed using these notations as:

$$\begin{aligned} A = [a_1, a_2] &= m(A) + r(A)[-1, 1] \\ &= \{x \in \mathbb{R} : \ |x - m(A)| \leq r(A)\} \end{aligned}$$

Let $A, B, C \in I(\mathbb{R})$. All intervals satisfy the following properties: $A + C = C + A$, $AB = BA$, $A + (B + C) = (A + B) + C$, $(AB)C = A(BC)$, $0 + A = A + 0 = A$ where $0 = [0, 0]$, $1A = A$, $A + (-A) = A - A$.

Let $A = [a_1, a_2] \in I(\mathbb{R})$ be given. Midpoints of symmetric intervals are 0. The one of the important problem of intervals that $A + (-A) = A - A = 0$ my not be satisfied for any interval. For example, let $A = [-1,2]$, then $-A = [-2,1]$ and $A - A = A + (-A) = [-1,2] + [-2,1] = [-3,3] \neq [0,0] = 0$. So, there is not the additive inverse of any interval, $(I(\mathbb{R}), +, \cdot)$ isn't a linear vector space. $A - A = 0$ if and only if $a_1 = a_2$. So, we can say that this property is satisfied only for degenerate intervals. To solve this problem with intervals, we need a new difference of two intervals.

Definition 2.1. [14,15] Let us consider two members A and B of $I(\mathbb{R})$. Then, generalized Hukuhara difference (shorly, gH difference) of A and B is defined as:

$$A -_g B = C \Leftrightarrow \begin{cases} A = B + C \\ B = A + (-1)C. \end{cases} or$$

gH difference have the following properties: Let $A, B \in I(\mathbb{R})$,

- $A -_g A = 0$
- $(A + B) -_g B = A$
- $A -_g (A - B) = B$
- $A -_g (A + B) = -B$
- $A -_g B$ exists every time
- $A -_g B = [\min\{a_1 - b_1, a_2 - b_2\}, \max\{a_1 - b_1, a_2 - b_2\}]$ where $A = [a_1, a_2]$ and $B = [b_1, b_2]$
- $t(A -_g B) = tA -_g tB$ for all $t > 0$ [14].

Because there is not a natural order relation on intervals as real number, we use the order relation to compare the intervals.

Definition 2.2. [1,2,6-11,12,16] Let $A, B \in I(\mathbb{R})$, $A = [a_1, a_2]$ and $B = [b_1, b_2]$. Then,

(i) $A \preceq_l B$ if and only if $a_1 \leq b_1$
(ii) $A \preceq_r B$ if and only if $a_2 \leq b_2$
(iii) $A \preceq_s B$ if and only if $a_2 \leq b_1$
(iv) $A \preceq_{lr} B$ if and only if $a_1 \leq b_1$ and $a_2 \leq b_2$
(v) $A \preceq_m B$ if and only if $A_m \leq B_m$
(vi) $A \preceq_r B$ if and only if $B_r \leq A_r$
(vii) $A \preceq_{mr} B$ if and only if $A_m \leq B_m$ and $B_r \leq A_r$

The radius of an interval is characterized with confidence interval. The values of the numbers of an interval give the magnitude of the interval. Therefore, it makes sense to choose the last order relation to find the largest among the intervals. Order relations are compatible with positive scalar multiplication. That is, $A \preceq_* B$ implies $tA \preceq_* tB$ for all $t > 0$ and for all $A, B \in I(\mathbb{R})$ and $* \in \{l, r, lr, m, r, mr, s\}$.

Proposition 2.2. $\preceq_{mr}$ is compatible with positive scalar multiplication.

Proof: Assume that $A = [a_1, a_2]$ and $B = [b_1, b_2]$ are two intervals, t is a positive number and $A \preceq_{mr} B$. We show that $tA \preceq_{mr} tB$. Since $A \preceq_{mr} B$, we have $A_m \leq B_m$ and $B_r \leq A_r$, that is,

$$\frac{a_1 + a_2}{2} \leq \frac{b_1 + b_2}{2} \tag{1}$$

and

$$\frac{b_2 - b_1}{2} \leq \frac{a_2 - a_1}{2}. \tag{2}$$

When we multiplty (1) and (2) by positive t, we obtain that $\frac{ta_1+ta_2}{2} \leq \frac{tb_1+tb_2}{2}$ and $\frac{tb_2-tb_1}{2} \leq \frac{ta_2-ta_1}{2}$. They imply that $(tA)_m \leq (tB)_m$ and $(tB)_r \leq (tA)_r$. Therefore, $tA \preceq_{mr} tB$.

Proposition 2.3. $\preceq_{mr}$ is compatible with addition, that is, if $A \preceq_{mr} B$ and $C \preceq_{mr} D$, then $A + C \preceq_{mr} B + D$ for all $A, B, C, D \in I(\mathbb{R})$.

Proof: Let $A \preceq_{mr} B$ and $C \preceq_{mr} D$ be satisfied for $A = [a_1, a_2], B = [b_1, b_2], C = [c_1, c_2]$ and $D = [d_1, d_2]$. Then, we have

$$\frac{a_1 + a_2}{2} \leq \frac{b_1 + b_2}{2}, \tag{3}$$

$$\frac{b_2 - b_1}{2} \leq \frac{a_2 - a_1}{2}, \tag{4}$$

$$\frac{c_1 + c_2}{2} \leq \frac{d_1 + d_2}{2}, \tag{5}$$

$$\frac{d_2 - d_1}{2} \leq \frac{c_2 - c_1}{2}. \tag{6}$$

From (3) and (5), (4) and (6) we get

$$\frac{a_1 + c_1 + a_2 + c_2}{2} \leq \frac{b_1 + d_1 + b_2 + d_2}{2}$$

and

$$\frac{b_2 + d_2 - b_1 - d_1}{2} \leq \frac{a_2 + c_2 - a_1 - c_1}{2}.$$

Therefore, $A + C \preceq_{mr} B + D$ satisfies.

The strictly version of this order relations are defined as:

Definition 2.3. Let $A, B \in I(\mathbb{R})$ and $* \in \{l, r, lr, s, m, r, mr\}$. Then,

$$A \prec_* B \Leftrightarrow A \preccurlyeq_* B \text{ and } A \neq B.$$

One can easily see that $\prec_*$ implies $\preccurlyeq_*$ for $* \in \{l, r, lr, m, r, mr, s\}$. Then, if $A \prec_* B$, then $A \preccurlyeq_* B$ for some $A, B \in I(\mathbb{R})$. Also, $\prec_*$ has some properties like $\preceq_*$.

Proposition 2.4. $\prec_{mr}$ is also compatible with positive scalar multiplication.

Proof: It can be obtained in a similar way the proof of Proposition 2.2.

Proposition 2.5. $\prec_{mr}$ is compatible with addition, that is, if $A \prec_{mr} B$ and $C \prec_{mr} D$, then $A + C \prec_{mr} B + D$ for all $A, B, C, D \in I(\mathbb{R})$.

Proof: It can be obtained in a similar way the proof of Proposition 2.3.

Proposition 2.6. Let $A \preceq_{mr} B$ be satisfied for some $A, B \in I(\mathbb{R})$. Then, $A -_g B \preceq_{mr} 0$. That is, if $A \preceq_{mr} B$, then $A -_g B \preceq_{mr} 0$ for $A, B \in I(\mathbb{R})$.

Proof: Assume that $A \preceq_{mr} B$ for some $A = [a_1, a_2], B = [b_1, b_2] \in I(\mathbb{R})$. Then, we have $A_m \leq B_m$ and $B_r \leq A_r$, that is,

$$\frac{a_1+a_2}{2} \leq \frac{b_1+b_2}{2} \Rightarrow a_1+a_2 \leq b_1+b_2 \tag{7}$$

and

$$\frac{b_2-b_1}{2} \leq \frac{a_2-a_1}{2} \Rightarrow b_2-b_1 \leq a_2-a_1. \tag{8}$$

From (8) we have

$$a_1-b_1 \leq a_2-b_2. \tag{9}$$

Then, $A{-_g}B = [\min\{a_1-b_1, a_2-b_2\}, \max\{a_1-b_1, a_2-b_2\}] = [a_1-b_1, a_2-b_2]$ from (9). Using (7) and (9) we get $a_2-b_2+a_1-b_1 \leq 0$ and $0 \leq a_2-b_2-a_1+b_1$, respectively. They imply that $\left(A{-_g}B\right)_m \leq 0$ and $0 \leq \left(A{-_g}B\right)_r$. Therefore, $A{-_g}B \preceq_{mr} 0$.

The following example illustrate Proposition 2.6.

Example 2.1. Let $A=[2,4]$ and $B=[5,6]$. Since $A_m = 3 \leq B_m = \frac{11}{2}$ and $B_r = \frac{1}{2} \leq A_r = 1$, $A \preccurlyeq_{mr} B$. $A{-_g}B = [\min\{2-5, 4-6\}, \max\{2-5, 4-6\}] = [-3,-2]$. Since $\left(A{-_g}B\right)_m = -\frac{5}{2} \leq 0$ and $0 \leq \left(A{-_g}B\right)_r = \frac{1}{2}$, $A{-_g}B \preccurlyeq_{mr} 0$.

The converse implication of Proposition 2.6 may not be true. Then, we can find two intervals that although they satisfy $A{-_g}B \preceq_{mr} 0$, $A \npreccurlyeq_{mr} B$. For example, let $A=[2,4]$ and $B=[2,5]$. Then, $A{-_g}B = [\min\{2-2, 4-5\}, \max\{2-2, 4-5\}] = [-1,0]$. Also, since $\left(A{-_g}B\right)_m = -\frac{1}{2} \leq 0$ and $0 \leq \left(A{-_g}B\right)_r = \frac{1}{2}$, $A{-_g}B \preceq_{mr} 0$. Moreover, since $A_m = 3 \leq B_m = \frac{7}{2}$ and $B_r = \frac{3}{2} \nleq A_r = 1$, $A \npreccurlyeq_{mr} B$.

Proposition 2.7. Let $A \prec_{mr} B$ be satisfied for some $A, B \in I(\mathbb{R})$. Then, $A -_g B \prec_{mr} 0$. That is, if $A \prec_{mr} B$, then $A -_g B \prec_{mr} 0$ for $A, B \in I(\mathbb{R})$.

Proof: It can be obtained like the proof of Proposition 2.6.

Because there are some order relations on $I(\mathbb{R})$, we can compare the intervals. Therefore, the following definition is used to find the efficient or extremum intervals of a family.

Definition 2.4. Let $* \in \{l, r, lr, s, m, r, mr\}$, $K \subseteq I(\mathbb{R})$ and $A \in K$ be given. Then, interval A is called

(i) minimal interval of K if there is no any interval $B \in K$ such that different from A and $B \preccurlyeq_* A$,

(ii) maximal interval of K if there is no any interval $B \in K$ such that different from A and $A \preccurlyeq_* B$,

(iii) weak minimal interval of K if there is no any interval $B \in K$ such that $B \prec_* A$,

(iv) weak maximal interval of K if there is no any interval $B \in K$ such that $A \prec_* B$,

(v) strongly minimal interval of K if $A \preccurlyeq_* B$ for all $B \in K$,

(vi) strongly maximal interval of K if $B \preccurlyeq_* A$ for all $B \in K$,

(vii) strictly minimal interval of K if $A \prec_* B$ for all $B \in K \,/\, \{A\}$,

(viii) strictly maximal interval of K if $B \prec_* A$ for all $B \in K \,/\, \{A\}$.

If an interval is a strictly minimal (or strictly maximal) interval of a set, then it is also a strongly minimal (or strongly maximal) interval of same set. If an interval is a strongly minimal (or strongly maximal) interval of a set, then it is also a minimal (or maximal) interval of same set. Similarly, if an interval is a minimal (or maximal) interval of a set, then it is also a weak minimal (or weak maximal) interval of same set.

It is not easy to find the extremum elements of a family using the above definition. In order to determine that an interval is extremum, it is necessary to know the relations of this interval with all other intervals according to the given order relation. The following example is given to explain this better.

Example 2.2. Let $K = \{[-1,2],[0,4],[-4,-2],[-3,0]\}$ and $\preceq_{lr}$ be given. The extremum intervals of K are found using the following:

- For $[-1,2]$ and $[0,4]$:

$-1 \leq 0$ and $2 \leq 4 \Longrightarrow [-1,2] \preceq_{lr} [0,4]$

$0 \nleq -1$ and $4 \nleq 2 \Longrightarrow [0,4] \npreceq_{lr} [-1,2]$

- For $[-1,2]$ and $[-4,-2]$:

$-1 \nleq -4$ and $2 \nleq -2 \Longrightarrow [-1,2] \npreceq_{lr} [-4,-2]$

$-4 \leq -1$ and $-2 \leq 2 \Longrightarrow [-4,-2] \preceq_{lr} [-1,2]$

- For $[-1,2]$ and $[-3,0]$:

$-1 \nleq -3$ and $2 \nleq 0 \Longrightarrow [-1,2] \npreceq_{lr} [3,0]$

$-3 \leq -1$ and $0 \leq 2 \Longrightarrow [-3,0] \preceq_{lr} [-1,2]$

- For $[0,4]$ and $[-4,-2]$:

$0 \nleq -4$ and $4 \nleq -2 \Longrightarrow [0,4] \npreceq_{lr} [-4,-2]$

$-4 \leq 0$ and $-2 \leq 4 \Longrightarrow [-4,-2] \preceq_{lr} [0,4]$

- For $[0,4]$ and $[-3,0]$:

$0 \nleq -3$ and $4 \nleq 0 \Longrightarrow [0,4] \npreceq_{lr} [-3,0]$

$-3 \leq 0$ and $0 \leq 4 \Longrightarrow [-3,0] \preceq_{lr} [0,4]$

- For $[-4,-2]$ and $[-3,0]$:

$-4 \leq -3$ and $-2 \leq 0 \Longrightarrow [-4,-2] \preceq_{lr} [-3,0]$

$-3 \nleq -4$ and $0 \nleq -2 \Longrightarrow [-3,0] \npreceq_{lr} [-4,-2]$

$[-4,-2]$ is minimal interval or minimal element of K because there is no an interval on K such that less than $[-4,-2]$. $[0,4]$ is maximal interval or maximal element of K because there is not an interval on K such that greater than $[0,4]$. Similarly, weak minimal and weak maximal element(s) can be find using the similar way.

An Hausdorff metric for intervals defined by Neumaier (1990) as: Let $A = [a_1,a_2], B = [b_1,b_2] \in I(\mathbb{R})$ be given. Then, $d_I : I(\mathbb{R}) \times I(\mathbb{R}) \to \mathbb{R}$ is a metric on interval numbers defined as:

$$d_I(A,B) \coloneqq \max\{|a_1 - b_1|, |a_2 - b_2|\}.$$

Then, $I(\mathbb{R})$ is a metric space with d_I [11,12].

The function $||\cdot||: I(\mathbb{R}) \rightarrow \mathbb{R}$ defined as $\|A\| = max\,\{|a_1|, |a_2|\}$ for all $A = [a_1, a_2] \in I(\mathbb{R})$ is a norm on $I(\mathbb{R})$. Therefore,
$d_I(A, B) = \left|A -_g B\right| = \max\{|a_1 - b_1|, |a_2 - b_2|\}$ for all $A = [a_1, a_2], B = [b_1, b_2] \in I(\mathbb{R})$ [5,15].

Now, interval sequences and convergence of them are considered. Let $(A_n)_{n\in\mathbb{N}}$ and A be intervals for all $n \in \mathbb{N}$. When n goes to infinity closed interval sequence A_n converges to interval A, and denoted by

$$\lim_{n\to\infty} A_n = A.$$

That is, for ever $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d_I(A, A_n) < \varepsilon$ for all $n > n_0$ [16].

Proposition 2.8. Let $A_n := [(a_1)_n, (a_2)_n]$ and $A = [a_{1_0}, a_{2_0}]$ be closed intervals for all $n \in \mathbb{N}$. Then, $\lim\limits_{n\to\infty} A_n = A$ if and only if $\lim\limits_{n\to\infty} (a_1)_n = a_{1_0}$ and $\lim\limits_{n\to\infty} (a_2)_n = a_{2_0}$ [16].

2.2. INTERVAL FUNCTIONS

Interval-valued functions and some properties are recalled in this part.
Let $X \subseteq \mathbb{R}^n$ be a nonempty set, then an interval-valued function or interval function $F: X \rightarrow I(\mathbb{R})$ is defined as $F(x) = [f_1(x), f_2(x)]$ for all $x \in X$, where $f_1: X \rightarrow \mathbb{R}$ and $f_2: X \rightarrow \mathbb{R}$ are real-valued functions, and called left and right end functions, respectively. It is obvious that $f_1(x) \leq f_2(x)$ for all $x \in X$ from the definition of intervals. It is clear that the interval functions are a generalization of scalar-valued functions. We will assume that $X \subseteq \mathbb{R}^n$ in the rest of work.

Now, some definitions and calculus rules are examined for interval functions.

Definition 2.5. Let X be a convex set and $F: X \rightarrow I(\mathbb{R})$ be an interval function. Then, F is called mr-convex at $\bar{x} \in X$ if

$$F(t\bar{x} + (1-t)x) \preccurlyeq_{mr} tF(\bar{x}) + (1-t)F(x)$$

for all $t \in (0,1)$ and each $x \in X$. If F is mr-convex for all $x \in X$, then F is called mr-convex interval function.

Definition 2.6. Let X be a convex set and $F: X \to I(\mathbb{R})$ be an interval function. Then, F is called strictly mr-convex at $\bar{x} \in X$ if

$$F(t\bar{x} + (1-t)x) \prec_{mr} tF(\bar{x}) + (1-t)F(x)$$

for all $t \in (0,1)$ and each $x \in X$. If F is strictly mr-convex for all $x \in X$, then F is called strictly mr-convex interval function.

Proposition 2.9. Let X be a nonempty convex set and and $F: X \to I(\mathbb{R})$, $F(x) = [f_l(x), f_u(x)]$ for all $x \in X$, be an interval function.

The following properties are satisfied:

(i) If F is strictly mr-convex, then it is an mr-convex

(ii) F is mr-convex iff F_m is a convex function and F_r is a concave function.

(iii) If F is mr-convex, then f_l is a convex function.

Proof:

(i) It can be easily obtained using Definition 2.2, Definition 2.3, Definition 2.5 and Definition 2.6.

(ii) Let F be an mr-convex interval function. Then, we have,

$$F(tx + (1-t)y) \preccurlyeq_{mr} tF(x) + (1-t)F(y)$$

for all $k \in (0,1)$ and all $x, y \in X$. Then, we get

$$\begin{aligned}[f_l(tx + (1-t)y), f_u(tx + (1-t)y)] &\preccurlyeq_{mr} t[f_l(x), f_u(x)] \\ &+ (1-t)[f_l(y), f_u(y)] \\ &= [tf_l(x), tf_u(x)] + [(1-t)f_l(y), (1-t)f_u(y)] \\ &= [tf_l(x)+(1-t)f_l(y), tf_u(x) + (1-t)f_u(y)]\end{aligned}$$

From mr-order relation,

$$\frac{f_l(tx + (1-t)y) + f_u(tx + (1-t)y)}{2} \leq \frac{tf_l(x)+(1-t)f_l(y) + tf_u(x) + (1-t)f_u(y)}{2}$$

and

$$\frac{tf_u(x) + (1-t)f_u(y) - tf_l(x) - (1-t)f_l(y)}{2} \leq \frac{f_u(tx + (1-t)y) - f_l(tx + (1-t)y)}{2}.$$

So,

$$F_m(tx + (1-t)y) \leq tF_m(x) + (1-t)F_m(y)$$

and

$$F_r(tx + (1-t)y) \geq tF_r(x) + (1-t)F_r(y)$$

Therefore, F_m is a convex function on X and F_r is a concave function on X.

(iii) Let F be an mr-convex interval function. Then, we have

$$F(tx + (1-t)y) \preccurlyeq_{mr} tF(x) + (1-t)F(y)$$

for all $t \in (0,1)$ and all $x, y \in X$. Then, we get

$$\begin{aligned}[f_l(tx + (1-t)y), f_u(tx + (1-t)y)] &\preccurlyeq_{mr} t[f_l(x), f_u(x)] \\ &\quad + (1-t)[f_l(y), f_u(y)] \\ &= [tf_l(x), tf_u(x)] + [(1-t)f_l(y), (1-t)f_u(y)] \\ &= [tf_l(x) + (1-t)f_l(y), tf_u(x) + (1-t)f_u(y)].\end{aligned}$$

From mr-order relation,

$$\frac{f_l(tx + (1-t)y) + f_u(tx + (1-t)y)}{2} \leq \frac{tf_l(x) + (1-t)f_l(y) + tf_u(x) + (1-t)f_u(y)}{2} \tag{10}$$

and

$$\frac{tf_u(x) + (1-t)f_u(y) - tf_l(x) - (1-t)f_l(y)}{2} \leq \frac{f_u(tx + (1-t)y) - f_l(tx + (1-t)y)}{2}. \tag{11}$$

Addition (10) and (11) we obtain

$$f_l(tx + (1-t)y) \leq tf_l(x) + (1-t)f_l(y)$$

Therefore, f_l is a convex function on X.

Definition: 2.7. Let $F: X \to I(\mathbb{R})$ be interval function defined as: $F(x)$ for all $x \in X$ near x_0, except possibly at x_0 itself, if we can ensure that $F(x)$ is as closed as we want to interval M by taking x close enough to x_0, but not equal to x_0, then M is called limit of F as x approaches x_0, denoted by $\lim_{x \to x_0} F(x) = M$. Then, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_I(F(x), M) < \varepsilon$ for $\|x - x_0\| < \delta$.

Proposition 2.10. Let $F: X \to I(\mathbb{R})$ be interval function defined as $F(x) = [f_1(x), f_2(x)]$ for all $x \in X$ and $M = [m_1, m_2]$ be an interval. Then, $\lim_{x \to x_0} F(x) = M$ if and only if $\lim_{x \to x_0} f_1(x) = m_1$ and $\lim_{x \to x_0} f_2(x) = m_2$ [16].

Definition 2.8. Let $F: X \to I(\mathbb{R})$ be an interval function and $x_0 \in X$. If

$$\lim_{x \to x_0} F(x) = F(x_0)$$

or equally, $\lim_{x \to x_0} \left(F(x) -_g F(x_0)\right) = 0$. Then, F is called gH–continuous at x_0.

Definition 2.9. Let $X \subseteq \mathbb{R}$ be nonempty set, $F: X \to I(\mathbb{R})$ be an interval function and $x_0 \in X$ be given. The gH–continuinity of F is defined at x_0 as: If for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\left\|x - x_0\right\| < \delta \Longrightarrow d_I\big(F(x_0 + h), F(x_0)\big) < \epsilon$ [2].

Proposition 2.11. Let $X \subseteq \mathbb{R}$ be nonempty set, $F: X \to I(\mathbb{R})$ be an interval function, it is defined $F(x) \coloneqq [f_1(x), f_2(x)]$ for all $x \in X$ and $x_0 \in X$ be given. Then, F is gH–continuous at x_0 iff f_1 and f_2 are continuous at same point [16].

Example 2.4. Let's take into account the following interval function $F: \mathbb{R} \to I(\mathbb{R})$ defined as $F(x) = \big[[|x|], x + 2\big]$ for all $x \in \mathbb{R}$, where $[|\cdot|]$ is exact–value or the greatest integer function. Now, examine the limit and gH–continuity of F at any point in the domain.

For example, let's check the point 2:

$\lim_{x \to 2^+} [\![x]\!] = 2$ *and* $\lim_{x \to 2^-} [|x|] = 1$, so there is no limit of $[|x|]$ at 2.

Since $\lim_{x \to 2^+} x + 2 = \lim_{x \to 2^-} x + 2 = 4$, $\lim_{x \to 2} x + 2 = 4$.

Then, there is no limit of F at 2. Therefore, F is not continuous at 2.

Definition 2.10. Let $F: X \to I(\mathbb{R})$ be an interval function and $x_0 \in X$. If $x_0 + h \in X$ for all enough small $h \in \mathbb{R}^n$ and the following limit

$$\lim_{h \to 0} \frac{F(x_0 + h) - F(x_0)}{h}$$

exists at x_0, then F is called GH-differentiable at x_0, and

$$F'(x_0) := \lim_{h \to 0} \frac{F(x_0 + h) - F(x_0)}{h}$$

is called the GH-derivative of F at x_0.

Definition 2.11. [15] Let $F: X \to I(\mathbb{R})$ be an interval function and $x_0 \in X$. If $x_0 + h \in X$ for all enough small $h \in \mathbb{R}^n$ and the following limit

$$\lim_{h \to 0} \frac{F(x_0 + h) -_g F(x_0)}{h}$$

exists at x_0, then F is called gH-differentiable at x_0, and

$$F'_H(x_0) := \lim_{h \to 0} \frac{F(x_0 + h) -_g F(x_0)}{h}$$

is called gH-derivative of F at x_0.

Definition 2.12. [5] Let $F: X \to I(\mathbb{R})$ be an interval function, $x_0 \in X$ and $h \in \mathbb{R}^n$. If the following limit

$$\lim_{h \to 0^+} \frac{F(x_0 + \lambda h) -_g F(x_0)}{\lambda}$$

exists at x_0, then F is called gH-directional differentiable at x_0, and

$$F'_h(x_0) := \lim_{h \to 0^+} \frac{F(x_0 + \lambda h) -_g F(x_0)}{\lambda}$$

is called gH-directional derivative of F at x_0 in the direction h.

Now, we will find the derivatives of an interval function in the next example. We will show that the gH-derivative of F may not be equal to GH-derivative of F at any point x_0. This situation illustrated in the next example.

Example 2.3. Let $F: [0,2\pi] \to I(\mathbb{R})$ be defined as:

$$F(x) = \begin{cases} [\sin(x), \cos(x)] & 0 \le x \le \dfrac{\pi}{4} \ or \ \dfrac{5\pi}{4} < x \le 2\pi \\ [\cos(x), \sin(x)] & \dfrac{\pi}{4} < x \le \dfrac{5\pi}{4} \end{cases}$$

Some image sets of interval function F are given in Figure 2.1.

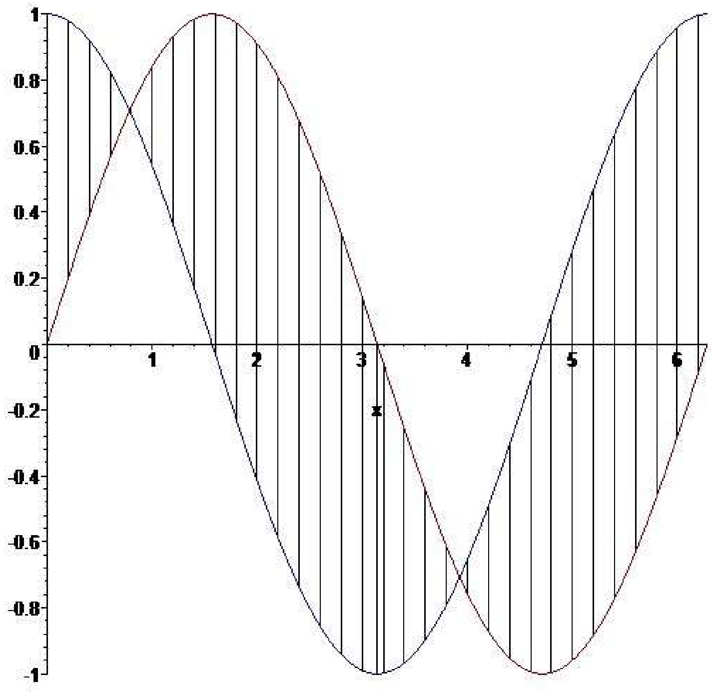

Figure 2.1: Image sets of interval function F defined in Example 2.2.

Then, the gH-derivative of F at $x_0 = \frac{\pi}{2}$ is

$$\begin{aligned} F'_H\left(\frac{\pi}{2}\right) &= \lim_{h\to 0} \frac{F\left(\frac{\pi}{2}+h\right) -_g F\left(\frac{\pi}{2}\right)}{h} \\ &= \lim_{h\to 0} \frac{\left[\cos\left(\frac{\pi}{2}+h\right), \sin\left(\frac{\pi}{2}+h\right)\right] -_g \left[\cos\left(\frac{\pi}{2}\right), \sin\left(\frac{\pi}{2}\right)\right]}{h} \\ &= \lim_{h\to 0} \frac{\left[\cos\left(\frac{\pi}{2}+h\right), \sin\left(\frac{\pi}{2}+h\right)\right] -_g [0,1]}{h} \\ &= \lim_{h\to 0} \frac{\left[\min\left\{\cos\left(\frac{\pi}{2}+h\right), \sin\left(\frac{\pi}{2}+h\right)-1\right\}, \max\left\{\cos\left(\frac{\pi}{2}+h\right), \sin\left(\frac{\pi}{2}+h\right)-1\right\}\ \right]}{h} \end{aligned}$$

If $h > 0$, then

$$\lim_{h\to 0^+} \frac{\left[\min\left\{\cos\left(\frac{\pi}{2}+h\right), \sin\left(\frac{\pi}{2}+h\right)-1\right\}, \max\left\{\cos\left(\frac{\pi}{2}+h\right), \sin\left(\frac{\pi}{2}+h\right)-1\right\}\right]}{h}$$

$$= \lim_{h\to 0^+} \left[\min\left\{\cos\left(\frac{\pi}{2}+h\right)/h, \sin\left(\frac{\pi}{2}+h\right)/h - 1/h\right\}, \max\left\{\cos\left(\frac{\pi}{2}+h\right)/h, \sin\left(\frac{\pi}{2}+h\right)/h - 1/h\right\}\right]$$

$$=[\min\{-1,0\}, \max\{-1,0\}] = [-1,0]$$

If $h < 0$, then

$$\lim_{h\to 0^-} \frac{\left[\min\left\{\cos\left(\frac{\pi}{2}+h\right), \sin\left(\frac{\pi}{2}+h\right)-1\right\}, \max\left\{\cos\left(\frac{\pi}{2}+h\right), \sin\left(\frac{\pi}{2}+h\right)-1\right\}\right]}{h}$$

$$= \lim_{h\to 0^-} -\left[\min\left\{-\cos\left(\frac{\pi}{2}+h\right)/h, -\frac{\sin\left(\frac{\pi}{2}+h\right)}{h} + 1/h\right\}, \max\left\{-\cos\left(\frac{\pi}{2}+h\right)/h, -\frac{\sin\left(\frac{\pi}{2}+h\right)}{h} + 1/h\right\}\right]$$

$$=-[\min\{1,0\}, \max\{1,0\}] = -[0,1] = [-1,0]$$

Then, F is gH-differentiable and $F'_H\left(\frac{\pi}{2}\right) = [-1,0]$.

Now, let's calculate the GH-derivative of F at $x_0 = \frac{\pi}{2}$.

$$F'\left(\frac{\pi}{2}\right) = \lim_{h\to 0} \frac{F\left(\frac{\pi}{2}+h\right) - F\left(\frac{\pi}{2}\right)}{h} = \lim_{h\to 0} \frac{\left[\cos\left(\frac{\pi}{2}+h\right), \sin\left(\frac{\pi}{2}+h\right)\right] - \left[\cos\left(\frac{\pi}{2}\right), \sin\left(\frac{\pi}{2}\right)\right]}{h}$$

$$= \lim_{h\to 0} \frac{\left[\cos\left(\frac{\pi}{2}+h\right), \sin\left(\frac{\pi}{2}+h\right)\right] - [0,1]}{h}$$

$$= \lim_{h\to 0} \frac{\left[\cos\left(\frac{\pi}{2}+h\right) - 1, \sin\left(\frac{\pi}{2}+h\right)\right]}{h}$$

If $h > 0$, then

$$F'\left(\frac{\pi}{2}\right) = \lim_{h\to 0} \frac{\left[\cos\left(\frac{\pi}{2}+h\right) - 1, \sin\left(\frac{\pi}{2}+h\right)\right]}{h} = \lim_{h\to 0^+} \left[\left(\cos\left(\frac{\pi}{2}+h\right) - 1\right)/h, \sin\left(\frac{\pi}{2}+h\right)/h\right]$$

Because $\lim_{h\to 0^+} \left(\cos\left(\frac{\pi}{2}+h\right) - 1\right)/h = -\infty$, there is no GH-derivative of F at $x_0 = \frac{\pi}{2}$.

Note that GH-derivative may be different from gH-derivative. If F is gH-differentiable function, then we can see two cases for gH-differentiable at any point $x \in int(X)$:

$$F'_H(x) = [f_1'(x), f_2{}'(x)]$$

or

$$F'_H(x) = [f_2'(x), f_1{}'(x)].$$

Theorem 2.1. Let gH-differentiable interval function $F: X \to I(\mathbb{R})$ be defined as $F(x) = [f_1(x), f_2(x)]$ for all $x \in X$, where $f_1, f_2: [a, b] \to \mathbb{R}$ real functions. Then, f_1 and f_2 are also differentiable and

$$F'_H(x_0) = [\min\{f_1'(x_0), f_2{}'(x_0)\}, \max\{f_1'(x_0), f_2{}'(x_0)\}]$$

satisfies for all $x_0 \in (a, b)$ [3,15].

Definition 2.13. [12,15] Let interval function $F: [a, b] \to I(\mathbb{R})$ be defined as $F(x) = [f_1(x), f_2(x)]$ for all $x \in [a, b]$. Integration of F is defined as

$$\int_a^b F(x)dx = \left[\int_a^b f_1(x)dx, \int_a^b f_2(x)dx\right].$$

Example 2.6. Let's calculate the integral of F defined Example 2.3:

$$\int_0^{2\pi} F(x)dx = \int_0^{\frac{\pi}{4}} [\sin(x), \cos(x)]dx +$$

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} [\cos(x), \sin(x)]dx + \int_{\frac{5\pi}{4}}^{2\pi} [\sin(x), \cos(x)]dx$$

$$=\left[\int_0^{\frac{\pi}{4}} \sin(x)\, dx, \int_0^{\frac{\pi}{4}} \cos(x)dx\right] + \left[\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos(x)dx, \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin(x)dx\right] +$$

$$\left[\int_{\frac{5\pi}{4}}^{2\pi} \sin(x)dx, \int_{\frac{5\pi}{4}}^{2\pi} \cos(x)dx\right]$$

$$= \left[-\cos(x)\,\Big|_0^{\frac{\pi}{4}}, \sin(x)\,\Big|_0^{\frac{\pi}{4}}\right] + \left[\sin(x)\,\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}, -\cos(x)\,\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\right] + \left[-\cos(x)\,\Big|_{\frac{5\pi}{4}}^{2\pi}, \sin(x)\,\Big|_{\frac{5\pi}{4}}^{2\pi}\right]$$

$$= \left[-\cos\frac{\pi}{4} + \cos 0, \sin\frac{\pi}{4} - \sin 0\right] +$$

$$\left[\sin\frac{5\pi}{4} - \sin\frac{\pi}{4}, -\cos\frac{5\pi}{4} + \cos\frac{\pi}{4}\right] + \left[-\cos 2\pi + \cos\frac{5\pi}{4}, \sin 2\pi - \sin\frac{5\pi}{4}\right]$$

$$= \left[-\cos\frac{\pi}{4} + \cos 0 + \sin\frac{5\pi}{4} - \sin\frac{\pi}{4} - \cos 2\pi + \cos\frac{5\pi}{4}, \sin\frac{\pi}{4} - \sin 0 - \cos\frac{5\pi}{4} + \cos\frac{\pi}{4} + \sin 2\pi - \sin\frac{5\pi}{4}\right]$$

$$= \left[-2\sqrt{2}, 2\sqrt{2}\right].$$

PART 3

INTERVAL OPTIMIZATION AND SOLUTIONS

Extremum points of interval function and solution of interval-valued optimization will present in this chapter.

Standard interval-valued optimization or interval optimization (shortly, (IVP)) is given by

$$(IVP)\begin{cases}\min(max)\, F(x)\\ \quad x \in X\end{cases}$$

where nonempty set X is called constraint set and interval function $F: X \to I(\mathbb{R})$ is called objective function. The aim of (IVP) is to find the best points that give the maximum and minimum values of F on X.

3.1. SOLUTIONS OF INTERVAL OPTIMIZATIONS

Because there is no natural order relation on intervals, order relations are used to solve these problems. We will use the order relations in Definition 2.2 to solve the problems. Some of these order relations are partial order relation, we will use the following definition.

Definition 3.1. Let $X \subseteq \mathbb{R}^n$ be nonempty set, an interval function $F: X \to I(\mathbb{R})$ and $* \in \{l, r, lr, s, m, r, mr\}$ be given. Then, $x_0 \in X$ is called a

(i) minimal solution of (IVP) if there is no any $x \in X$ such that the value of x_0 under F is different from $F(x_0)$ and $F(x) \preccurlyeq_* F(x_0)$,

(ii) maximal solution of (IVP) if there is no any $x \in X$ such that the value of x_0 under F is different from $F(x_0)$ and $F(x_0) \preccurlyeq_* F(x)$,

(iii) weak minimal solution of (IVP) if there is no any $x \in X$ such that $F(x) \prec_* F(x_0)$,

(iv) weak maximal solution of (IVP) if there is no any $x \in X$ such that $F(x_0) \prec_* F(x)$,

(v) strongly minimal solution of (IVP) if $F(x_0) \preccurlyeq_* F(x)$ for all $x \in X$,

(vi) strongly maximal solution of (IVP) if $F(x) \preccurlyeq_* F(x_0)$ for all $x \in X$,

(vii) strictly minimal solution of (IVP) if $F(x_0) \prec_* F(x)$ for all $x \in X \,/\, \{x_0\}$,

(viii) strictly maximal solution of (IVP) if $F(x) \prec_* F(x_0)$ for all $x \in X \,/\, \{x_0\}$.

If a point is a strictly solution of the problem, then it is also a strongly solution of the problem. If a point is a strongly solution of the problem, then it is also a solution of the problem. Moreover, if a point is a solution of the problem, then it is also a weak solution. Therefore, we obtain the following relation:

$$\begin{matrix}\text{Strictly}\\ \text{Solution}\end{matrix} \Longrightarrow \begin{matrix}\text{Strongly}\\ \text{Solution}\end{matrix} \Longrightarrow Solution \Longrightarrow \begin{matrix}Weak\\ Solution\end{matrix}$$

Note that while a point is a solution of a problem with respect to (shorty, wrt) an order relation, same point may not be a solution of the problem wrt another order relation.

Example 3.1. Let interval function $F: [-1,1] \to I(\mathbb{R})$ be defined as $F(x) = [x^2, |x|]$ for all $x \in [-1,1]$. Let's regard the following interval optimization

$$\begin{cases} \min F(x) \\ x \in [-1,1] \end{cases}$$

Image sets of F given in Figure 3.1.

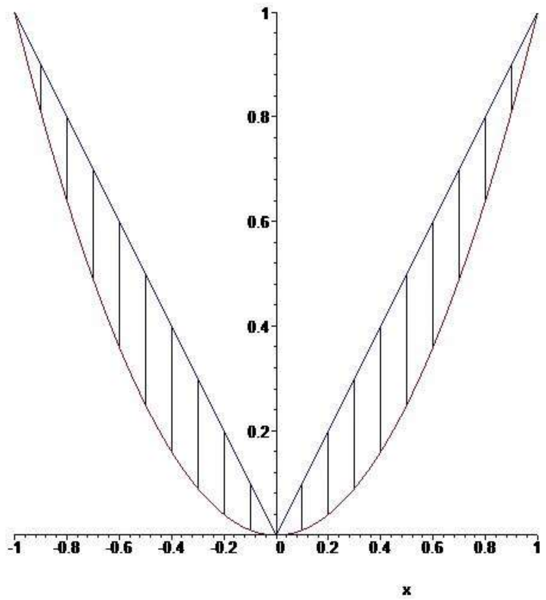

Figure 3.1: Image sets of $F(x) = [x^2, |x|]$ for all $x \in [-1,1]$.

We will solve this problem using the following order relations:

$\preccurlyeq_l$ **order relation:** 0 is minimal solution of the problem because there is no $x \in [-1,1]/\{0\}$ that $F(x) \preccurlyeq_l F(0)$. Really, assume that there exists a $x \in [-1,1]/\{0\}$ that

$$F(x) \preccurlyeq_l F(0) \Leftrightarrow [x^2, |x|] \preccurlyeq_l [0,0] \Leftrightarrow x^2 \leq 0$$

Then, there is no an $x \in [-1,1]/\{0\}$ such that $x^2 \leq 0$. So, 0 is a minimal solution. It is unique because of $F(0) \preccurlyeq_l F(x)$ for all $x \in [-1,1]$. 0 is weak minimal solution of the problem because there is no $x \in [-1,1]$ such $F(x) \prec_l F(0)$. Really, assume that there exists an $x \in [-1,1]$ that

$$F(x) \prec_l F(0) \Leftrightarrow [x^2, |x|] \prec_l [0,0] \Leftrightarrow x^2 < 0$$

We can not find an $x \in [-1,1]$ such that $x^2 < 0$. Then, 0 is a weak minimum solution of the problem. 0 is strongly minimal solution of the problem because $F(0) \preccurlyeq_l F(x)$ for all $x \in [-1,1]$. Really,

$$F(0) \preccurlyeq_l F(x) \Leftrightarrow [0,0] \preccurlyeq_l [x^2, |x|] \Leftrightarrow 0 \leq x^2$$

Also, 0 is strictly solution of the problem because $F(0) \prec_l F(x)$ for all $x \in [-1,1] / \{0\}$. Moreover, the problem has no solution other than 0 with respect to order relation $\preccurlyeq_l$.

$\preccurlyeq_{mr}$ **order relation:** Let's check the point 0: If 0 is a minimal solution of the problem, then we can not find a $x \in [-1,1]$ that $F(x) \preccurlyeq_{mr} F(0)$ and $F(x) \neq F(0)$. Assume the contrary that there is a $x \in [-1,1]$ such that $F(x) \preccurlyeq_{mw} F(0)$ and $F(x) \neq F(0)$. Then,

$$F(x) \preccurlyeq_{mr} F(0) \Leftrightarrow [x^2, |x|] \preccurlyeq_{mr} [0,0]$$

$$\Leftrightarrow \frac{x^2 + |x|}{2} \leq 0 \; and \; 0 \leq \frac{|x| - x^2}{2}$$

$$\Leftrightarrow x^2 + |x| \leq 0 \; and \; 0 \leq |x| - x^2$$

These two inequalities are satisfied for only 0. Then, there is not a $x \in [-1,1]$ such that $F(x) \neq F(0)$ and $F(x) \preccurlyeq_{mr} F(0)$. So, 0 is a solution of the problem.

Let's check the point 0; If 0 is a weak minimal solution of the problem, then we cannot find an $x \in [-1,1]$ that $F(x) \prec_{mr} F(0)$. Assume the contrary that there is an $x \in [-1,1]$ that $F(x) \prec_{mr} F(0)$. Then,

$$F(x) \prec_{mr} F(0) \Leftrightarrow [x^2, |x|] \prec_{mr} [0,0]$$

$$\Leftrightarrow \frac{x^2 + |x|}{2} \leq \; 0, 0 \leq \frac{|x| - x^2}{2} \; and \; F(x) \neq F(0)$$

$$\Leftrightarrow x^2 + |x| \leq 0, |x| - x^2 \; \geq 0 \; and \; and \; F(x) \neq F(0)$$

There is no a solution of these two inequalities, then 0 is a weak solution of the problem.

0 is strongly solution of the problem because $F(0) \preccurlyeq_{mr} F(x)$ for all $x \in [-1,1]$. Really,

$$F(0) \preccurlyeq_{mr} F(x) \Leftrightarrow [0,0] \preccurlyeq_{mr} [x^2, |x|]$$

$$\Leftrightarrow 0 \leq \frac{x^2 + |x|}{2} \; and \; \frac{|x| - x^2}{2} \leq 0$$

$$\Leftrightarrow x^2 + |x| \geq 0 \; and \; |x| - x^2 \; \leq 0$$

The solution of these two inequalities is only 0, that is $F(0) \preccurlyeq_{mr} F(x)$ is satisfied for only 0. Since $F(0) \not\preccurlyeq_{mr} F(x)$ all $x \in [-1,1]$, 0 is not strongly minimum solution of the problem. Similarly, we can show that 0 is not strictly solution of the problem.

Example 3.2. Let interval function $F: [0,2\pi] \to I(\mathbb{R})$ be defined as

$$F(x) = \begin{cases} [\sin(x), \cos(x)] & ; 0 \le x \le \dfrac{\pi}{4} \ or \ \dfrac{5\pi}{4} < x \le 2\pi \\ [\cos(x), \sin(x)] & ; \dfrac{\pi}{4} < x \le \dfrac{5\pi}{4} \end{cases} \tag{12}$$

Consider the following interval optimization

$$\begin{cases} \max F(x) \\ x \in [0,2\pi] \end{cases}$$

Image of F are given in Figure 3.2.

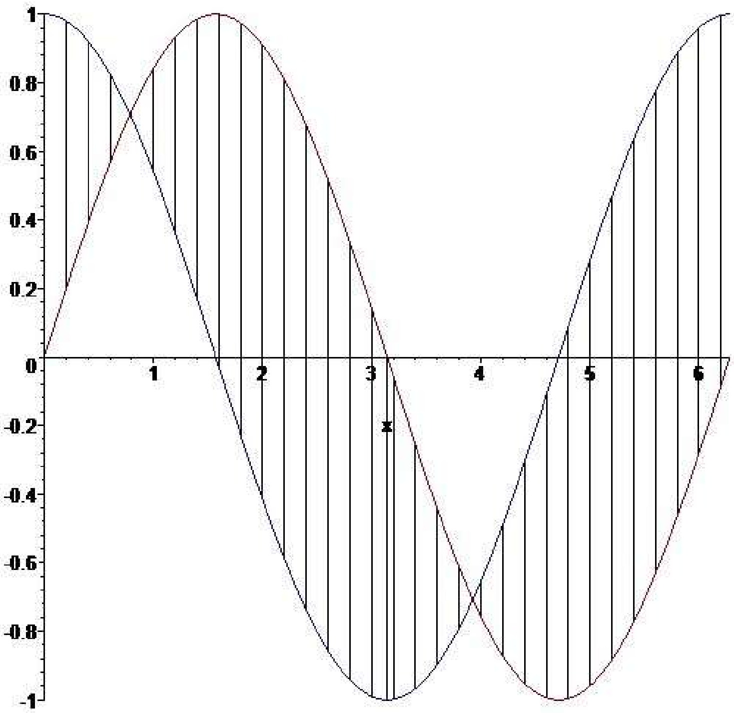

Figure 3.2. Image sets of F defined in Example 3.2.

This problem will be solved using different order relations:

$\preccurlyeq_{lr}$ **order relation:** Let's check the point $x_0 = \frac{\pi}{4}$. If it is a maximal solution of (12), then there is not exist an $x \in [0,2\pi]$ that $F\left(\frac{\pi}{4}\right) \neq F(x)$ and $F\left(\frac{\pi}{4}\right) \preccurlyeq_{lr} F(x)$. Assume the contrary that there exists an $x \in [0,2\pi]$ that $F\left(\frac{\pi}{4}\right) \neq F(x)$ and $F\left(\frac{\pi}{4}\right) \preccurlyeq_{lr} F(x)$ is satisfied. Then,

For $x \in \left[0, \frac{\pi}{4}\right] \cup \left(\frac{5\pi}{4}, 2\pi\right]$,

$$\begin{aligned} F\left(\frac{\pi}{4}\right) \preccurlyeq_{lr} F(x) \quad &\Leftrightarrow \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] \preccurlyeq_{lr} [\sin x, \cos x] \\ &\Leftrightarrow \frac{\sqrt{2}}{2} \leq \sin x \ \ and \ \frac{\sqrt{2}}{2} \leq \cos x \\ &\Leftrightarrow x = \frac{\pi}{4} \end{aligned}$$

For $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]$,

$$\begin{aligned} F\left(\frac{\pi}{4}\right) \preccurlyeq_{lr} F(x) \quad &\Leftrightarrow \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] \preccurlyeq_{lr} [\cos x, \sin x] \\ &\Leftrightarrow \frac{\sqrt{2}}{2} \leq \cos x \ \ and \ \frac{\sqrt{2}}{2} \leq \sin x \\ &\Leftrightarrow There\ is\ no\ solution \end{aligned}$$

Because there is no an $x \in [0,2\pi]$ that $F\left(\frac{\pi}{4}\right) \neq F(x)$ and $F\left(\frac{\pi}{4}\right) \preccurlyeq_{lr} F(x)$, $\frac{\pi}{4}$ is a maximal solution of the problem wrt order relation $\preccurlyeq_{lr}$.

$\frac{\pi}{2}$ is another solution the problem. Because we cannot find a $x \in [0,2\pi]$ such that $F(x) \neq F\left(\frac{\pi}{2}\right)$ and $F\left(\frac{\pi}{2}\right) \preccurlyeq_{lr} F(x)$. On the contrary to assumption, assume that there exists an $x \in [0,2\pi]$ that $F(x) \neq F\left(\frac{\pi}{2}\right)$ and $F\left(\frac{\pi}{2}\right) \preccurlyeq_{lr} F(x)$. Then,

For $x \in \left[0, \frac{\pi}{4}\right] \cup \left(\frac{5\pi}{4}, 2\pi\right]$,

$$\begin{aligned} F\left(\frac{\pi}{2}\right) \preccurlyeq_{lr} F(x) \quad &\Leftrightarrow [0,1] \preccurlyeq_{lr} [\sin x, \cos x] \\ &\Leftrightarrow 0 \leq \sin x \ \ and \ 1 \leq \cos x \\ &\Leftrightarrow x = 0 \ and \ x = 2\pi \end{aligned}$$

Because $F(0) = F(2\pi) = F\left(\frac{\pi}{2}\right)$, we didn't find an element such that $F(x) \neq F\left(\frac{\pi}{2}\right)$ and $F\left(\frac{\pi}{2}\right) \preccurlyeq_{lr} F(x)$ on $\left[0, \frac{\pi}{4}\right] \cup \left(\frac{5\pi}{4}, 2\pi\right]$.

For $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]$,

$$\begin{aligned} F\left(\frac{\pi}{2}\right) \preccurlyeq_{lr} F(x) \quad &\Leftrightarrow [0,1] \preccurlyeq_{lr} [\cos x, \sin x] \\ &\Leftrightarrow 0 \leq \cos x \ \ and \ 1 \leq \sin x \\ &\Leftrightarrow x = \frac{\pi}{2} \end{aligned}$$

There is no a solution different from $x = \frac{\pi}{2}$. So, we didn't find an $x \in [0,2\pi]$ that $F(x) \neq F\left(\frac{\pi}{2}\right)$ and $F\left(\frac{\pi}{2}\right) \preccurlyeq_{lr} F(x)$ on $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right]$. Therefore, $\frac{\pi}{2}$ is a solution of the problem wrt order relation $\preccurlyeq_{lr}$.

Assume that $\frac{\pi}{2}$ is weak maximal solution of (12). Then, there is no $x \in [0,2\pi]$ that $F\left(\frac{\pi}{2}\right) \prec_{lr} F(x)$. Assume the contrary that there exists an $x \in [0,2\pi]$ that $F\left(\frac{\pi}{2}\right) \prec_{lr} F(x)$.

For $x \in \left[0, \frac{\pi}{4}\right] \cup \left(\frac{5\pi}{4}, 2\pi\right]$,

$$F\left(\frac{\pi}{2}\right) \prec_{lr} F(x) \Leftrightarrow [0,1] \prec_{lr} [\sin x, \cos x]$$
$$\Leftrightarrow 0 \leq \sin x, \ \ 1 \leq \cos x \ \ and \ [\sin x, \cos x] \neq [0,1]$$

There is no solution of this system.

For $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]$,

$$F\left(\frac{\pi}{2}\right) \prec_{lr} F(x) \Leftrightarrow [0,1] \prec_{lr} [\cos x, \sin x]$$
$$\Leftrightarrow 0 \leq \cos x, \ \ 1 \leq \sin x \ \ and \ [\cos x, \sin x] \neq [0,1]$$

There is no solution of this system. Then, $\frac{\pi}{2}$ is a weak maximal solution of the problem.

Suppose that $\frac{\pi}{2}$ is strongly maximal solution of the problem. Then, $F(x) \preccurlyeq_{lr} F\left(\frac{\pi}{2}\right)$ must be satisfied for all $x \in [0,2\pi]$. So,

For $x \in \left[0, \frac{\pi}{4}\right] \cup \left(\frac{5\pi}{4}, 2\pi\right]$,

$$F(x) \preccurlyeq_{lr} F\left(\frac{\pi}{2}\right) \quad \Leftrightarrow \quad [\sin x, \cos x] \preccurlyeq_{lr} [0,1]$$
$$\Leftrightarrow \sin x \leq 0, \quad \cos x \leq 1$$
$$\Leftrightarrow x \in \left(\frac{5\pi}{4}, 2\pi\right] \cup \{0\}$$

For $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]$,

$$F(x) \preccurlyeq_{lr} F\left(\frac{\pi}{2}\right) \quad \Leftrightarrow \quad [\cos x, \sin x] \preccurlyeq_{lr} [0,1]$$
$$\Leftrightarrow \cos x \leq 0, \quad \sin x \leq 1$$
$$\Leftrightarrow x \in \left[\frac{\pi}{2}, \frac{5\pi}{4}\right]$$

Then, $F(x) \preccurlyeq_{lr} F\left(\frac{\pi}{2}\right)$ is satisfied for only $x \in \left(\frac{5\pi}{4}, 2\pi\right] \cup \left[\frac{\pi}{2}, \frac{5\pi}{4}\right] \cup \{0\}$. That is, $F(x) \preccurlyeq_{lr} F\left(\frac{\pi}{2}\right)$ is not satisfied for all $x \in [0,2\pi]$. For example, $F(x) \preccurlyeq_{lr} F\left(\frac{\pi}{2}\right)$ is not satisfied for $x = \frac{\pi}{4}$, Really,

$$F\left(\frac{\pi}{4}\right) \preccurlyeq_{lr} F\left(\frac{\pi}{2}\right) \Leftrightarrow \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] \preccurlyeq_{lr} [0,1]$$
$$\Leftrightarrow \frac{\sqrt{2}}{2} \leq 0 \text{ and } \frac{\sqrt{2}}{2} \leq 1$$

The last two inequalities do not satisfy. So, $F\left(\frac{\pi}{4}\right) \not\preccurlyeq_{lr} F\left(\frac{\pi}{2}\right)$ for all $x \in [0,2\pi]$. Therefore, $\frac{\pi}{2}$ is not strongly maximal solution.

Suppose that $\frac{\pi}{4}$ is strongly maximal solution of the problem. Then, $F(x) \preccurlyeq_{lr} F\left(\frac{\pi}{4}\right)$ must be satisfied for all $x \in [0,2\pi]$. So,

For $x \in \left[0, \frac{\pi}{4}\right] \cup \left(\frac{5\pi}{4}, 2\pi\right]$,

$$F(x) \preccurlyeq_{lr} F\left(\frac{\pi}{4}\right) \quad \Leftrightarrow \quad [\sin x, \cos x] \preccurlyeq_{lr} \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$$
$$\Leftrightarrow \sin x \leq \frac{\sqrt{2}}{2}, \quad \cos x \leq \frac{\sqrt{2}}{2}$$
$$\Leftrightarrow x \in \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right] \cup \left\{\frac{\pi}{4}\right\}$$

For $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]$,

$$F(x) \preccurlyeq_{lr} F\left(\frac{\pi}{4}\right) \quad \Leftrightarrow \quad [\cos x, \sin x] \preccurlyeq_{lr} \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$$
$$\Leftrightarrow \cos x \leq \frac{\sqrt{2}}{2}, \quad \sin x \leq \frac{\sqrt{2}}{2}$$
$$\Leftrightarrow x \in \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$$

Then, $F(x) \preccurlyeq_{lr} F\left(\frac{\pi}{4}\right)$ is satisfied for only $x \in \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right] \cup \left\{\frac{\pi}{4}\right\}$. That is, $F(x)$ $\preccurlyeq_{lr} F\left(\frac{\pi}{4}\right)$ is not satisfied for all $x \in [0,2\pi]$. For example, $F(x) \preccurlyeq_{lr} F\left(\frac{\pi}{4}\right)$ is not satisfied for $x = 0$. Really,

$$F(0) \preccurlyeq_{lr} F\left(\frac{\pi}{4}\right) \Leftrightarrow \;\; [0,1] \preccurlyeq_{lr} \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$$

$$\Leftrightarrow 0 \leq \frac{\sqrt{2}}{2} \text{ and } 1 \leq \frac{\sqrt{2}}{2}$$

The last two inequalities do not satisfy. So, $F(0) \not\preccurlyeq_{lr} F\left(\frac{\pi}{4}\right)$ for all $x \in [0,2\pi]$. Therefore, $\frac{\pi}{4}$ is not strongly maximal solution.

$\preccurlyeq_{mr}$ **order relation:** Let's check the point $x_0 = \frac{\pi}{4}$. If it is a solution of (12), there is not exist an $x \in [0,2\pi]$ such that $F\left(\frac{\pi}{4}\right) \neq F(x)$ and $F\left(\frac{\pi}{4}\right) \preccurlyeq_{mr} F(x)$. Assume the contrary that there exists an $x \in [0,2\pi]$ such that $F\left(\frac{\pi}{4}\right) \neq F(x)$ and $F\left(\frac{\pi}{4}\right) \preccurlyeq_{mr} F(x)$ are satisfied. Then,

For $x \in \left[0, \frac{\pi}{4}\right] \cup \left(\frac{5\pi}{4}, 2\pi\right]$,

$$\begin{aligned} F\left(\frac{\pi}{4}\right) \preccurlyeq_{mr} F(x) \quad &\Leftrightarrow \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] \preccurlyeq_{mr} [\sin x, \cos x] \\ &\Leftrightarrow \frac{\sqrt{2}}{2} \leq \frac{\sin x + \cos x}{2} \;\; and \;\; \frac{\cos x - \sin x}{2} \leq 0 \\ &\Leftrightarrow x = \frac{\pi}{4} \end{aligned}$$

For $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]$,

$$\begin{aligned} F\left(\frac{\pi}{4}\right) \preccurlyeq_{mr} F(x) \quad &\Leftrightarrow \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] \preccurlyeq_{mr} [\cos x, \sin x] \\ &\Leftrightarrow \frac{\sqrt{2}}{2} \leq \frac{\cos x + \sin x}{2} \;\; and \;\; \frac{\sin x - \cos x}{2} \leq 0 \\ &\Leftrightarrow There\ is\ no\ a\ solution \end{aligned}$$

Because there is no an $x \in [0,2\pi]$ such that $F\left(\frac{\pi}{4}\right) \neq F(x)$ and $F\left(\frac{\pi}{4}\right) \preccurlyeq_{mr} F(x)$, $\frac{\pi}{4}$ is a maximal solution of the problem wrt order relation $\preccurlyeq_{mr}$.

$\frac{\pi}{2}$ is not a solution the problem. Because we can find an $x \in [0,2\pi]$ such that $F(x) \neq F\left(\frac{\pi}{2}\right)$ and $F\left(\frac{\pi}{2}\right) \preccurlyeq_{mr} F(x)$. Really,

For $x \in \left[0,\frac{\pi}{4}\right] \cup \left(\frac{5\pi}{4}, 2\pi\right]$,

$$\begin{aligned} F\left(\frac{\pi}{2}\right) \preccurlyeq_{mr} F(x) \quad &\Leftrightarrow [0,1] \preccurlyeq_{mr} [\sin x, \cos x] \\ &\Leftrightarrow \frac{1}{2} \leq \frac{\sin x + \cos x}{2} \ and\ \frac{\cos x - \sin x}{2} \leq \frac{1}{2} \\ &\Leftrightarrow x \in \left[0,\frac{\pi}{4}\right] \end{aligned}$$

For $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]$,

$$\begin{aligned} F\left(\frac{\pi}{2}\right) \preccurlyeq_{mr} F(x) \quad &\Leftrightarrow [0,1] \preccurlyeq_{mr} [\cos x, \sin x] \\ &\Leftrightarrow \frac{1}{2} \leq \frac{\cos x + \sin x}{2} \ and\ \frac{\sin x - \cos x}{2} \leq \frac{1}{2} \\ &\Leftrightarrow x \in \left(\frac{\pi}{4},\frac{\pi}{2}\right] \end{aligned}$$

We can find an $x \in [0,2\pi]$ such that $F(x) \neq F\left(\frac{\pi}{2}\right)$ and $F\left(\frac{\pi}{2}\right) \preccurlyeq_{mr} F(x)$. That is, $F\left(\frac{\pi}{2}\right) \preccurlyeq_{mr} F(x)$ is not satisfied for all $x \in [0,2\pi]$. Therefore, $\frac{\pi}{2}$ is not a solution of the problem wrt order relation $\preccurlyeq_{mr}$.

$\frac{\pi}{2}$ is not a weak maximal solution of (12) because we can find an $x \in [0,2\pi]$ such that $F\left(\frac{\pi}{2}\right) \prec_{mr} F(x)$. Really, for example, $F\left(\frac{\pi}{2}\right) \prec_{mr} F(x)$ is satisfied for all $x \in \left(0,\frac{\pi}{2}\right)$.

Because of

$$F\left(\frac{\pi}{2}\right) \prec_{mr} F(x) \Leftrightarrow \left[\cos\frac{\pi}{2}, \sin\frac{\pi}{2}\right] \prec_{mr} F(x)$$

$$\Leftrightarrow [0,1] \prec_{mr} [\sin x, \cos x]$$

$$\Leftrightarrow \frac{1}{2} \leq \frac{\sin x + \cos x}{2}, \frac{\cos x - \sin x}{2} \leq \frac{1}{2} \; and \; [0,1] \neq [\sin x, \cos x]$$

$$\Leftrightarrow x \in \left(0, \frac{\pi}{2}\right)$$

Now, we will check $\frac{\pi}{4}$ for weak solution. We wait that it is a weak solution of (12) because a solution of a problem is also a weak solution.

If it is a solution of (12), there is not exist an $x \in [0,2\pi]$ such that $F\left(\frac{\pi}{4}\right) \prec_{mr} F(x)$.

Assume the contrary that there exists an $x \in [0,2\pi]$ that $F\left(\frac{\pi}{4}\right) \prec_{mr} F(x)$. Then,

For $x \in \left[0, \frac{\pi}{4}\right] \cup \left(\frac{5\pi}{4}, 2\pi\right]$,

$$F\left(\frac{\pi}{4}\right) \prec_{mr} F(x) \Leftrightarrow \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] \prec_{mr} [\sin x, \cos x]$$

$$\Leftrightarrow \frac{\sqrt{2}}{2} \leq \frac{\sin x + \cos x}{2}, \frac{\cos x - \sin x}{2} \leq 0 \; and \; \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] \neq [\sin x, \cos x]$$

Then, there is no a solution of this system.

For $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]$,

$$F\left(\frac{\pi}{4}\right) \prec_{mr} F(x) \Leftrightarrow \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] \prec_{mr} [\cos x, \sin x]$$

$$\Leftrightarrow \frac{\sqrt{2}}{2} \leq \frac{\cos x + \sin x}{2}, \frac{\sin x - \cos x}{2} \leq 0 \; and$$

$$\left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] \neq [\cos x, \sin x]$$

Then, there is no a solution of this system. Therefore, $\frac{\pi}{4}$ is a weak maximal solution of (12).

Because $\frac{\pi}{2}$ is not solution or weak solution of (12), we can say that it is not strongly or strictly solution of the problem.

Now, let's check the point $\frac{\pi}{4}$ for strongly maximal solution. If it is a strongly maximal solution of the problem, then $F(x) \preccurlyeq_{mr} F\left(\frac{\pi}{4}\right)$ must be satisfied for all $x \in [0,2\pi]$.

For $x \in \left[0, \frac{\pi}{4}\right] \cup \left(\frac{5\pi}{4}, 2\pi\right]$,

$$F(x) \preccurlyeq_{mr} F\left(\frac{\pi}{4}\right) \iff [\sin x, \cos x] \preccurlyeq_{mr} \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$$

$$\iff \frac{\sin x + \cos x}{2} \leq \frac{\sqrt{2}}{2} \;\; and \;\; 0 \leq \frac{\cos x - \sin x}{2}$$

$$\iff x \in \left[0, \frac{\pi}{4}\right] \cup \left(\frac{5\pi}{4}, 2\pi\right],$$

For $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]$,

$$F(x) \preccurlyeq_{mr} F\left(\frac{\pi}{4}\right) \iff [\cos x, \sin x] \preccurlyeq_{mr} \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$$

$$\iff \frac{\cos x + \sin x}{2} \leq \frac{\sqrt{2}}{2} \;\; and \;\; 0 \leq \frac{\sin x - \cos x}{2}$$

$$\iff x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]$$

Therefore, we have $F(x) \preccurlyeq_{mr} F\left(\frac{\pi}{4}\right)$ for all $x \in [0,2\pi]$, which implies that $\frac{\pi}{4}$ is a strongly solution of (12). Similarly, we can show that $F(x) \prec_{mr} F\left(\frac{\pi}{4}\right)$ for all $x \in [0,2\pi] \,/\, \left\{\frac{\pi}{4}\right\}$, which implies that $\frac{\pi}{4}$ is a strictly maximal solution of (12).

$\preccurlyeq_s$ **order relation:** Let's check the point $x_0 = \frac{\pi}{4}$. If it is a solution of (12), there is not exist an $x \in [0,2\pi]$ such that $F\left(\frac{\pi}{4}\right) \neq F(x)$ and $F\left(\frac{\pi}{4}\right) \preccurlyeq_s F(x)$. Assume the contrary that there exists an $x \in [0,2\pi]$ such that $F\left(\frac{\pi}{4}\right) \neq F(x)$ and $F\left(\frac{\pi}{4}\right) \preccurlyeq_s F(x)$ is satisfied. Then,

For $x \in \left[0, \frac{\pi}{4}\right] \cup \left(\frac{5\pi}{4}, 2\pi\right]$,

$$F\left(\frac{\pi}{4}\right) \preccurlyeq_s F(x) \quad \Leftrightarrow \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] \preccurlyeq_s [\sin x, \cos x]$$
$$\Leftrightarrow \frac{\sqrt{2}}{2} \leq \sin x$$
$$\Leftrightarrow x = \frac{\pi}{4}.$$

For $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]$,

$$F\left(\frac{\pi}{4}\right) \preccurlyeq_s F(x) \quad \Leftrightarrow \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] \preccurlyeq_s [\cos x, \sin x]$$
$$\Leftrightarrow \frac{\sqrt{2}}{2} \leq \cos x$$

There is no solution of the last inequalities. Then, because there is no an $x \in [0,2\pi]$ such that $F\left(\frac{\pi}{4}\right) \neq F(x)$ and $F\left(\frac{\pi}{4}\right) \preccurlyeq_s F(x)$, $\frac{\pi}{4}$ is a maximal solution of the problem wrt order relation $\preccurlyeq_s$.

$\frac{\pi}{2}$ is another solution the problem. Because we cannot find an $x \in [0,2\pi]$ such that $F(x) \neq F\left(\frac{\pi}{2}\right)$ and $F\left(\frac{\pi}{2}\right) \preccurlyeq_s F(x)$. On the contrary to assumption, assume that there exists an $x \in [0,2\pi]$ such that $F(x) \neq F\left(\frac{\pi}{2}\right)$ and $F\left(\frac{\pi}{2}\right) \preccurlyeq_s F(x)$. Then,

For $x \in \left[0, \frac{\pi}{4}\right] \cup \left(\frac{5\pi}{4}, 2\pi\right]$,

$$F\left(\frac{\pi}{2}\right) \preccurlyeq_s F(x) \quad \Leftrightarrow [0,1] \preccurlyeq_s [\sin x, \cos x]$$
$$\Leftrightarrow 1 \leq \sin x$$
$$\Leftrightarrow \textit{There is no a solution}$$

For $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]$,

$$F\left(\frac{\pi}{2}\right) \preccurlyeq_s F(x) \quad \Leftrightarrow [0,1] \preccurlyeq_s [\cos x, \sin x]$$
$$\Leftrightarrow 1 \leq \cos x$$

There is no a solution of the last inequality. So, we didn't find a $x \in [0,2\pi]$ that $F(x) \neq F\left(\frac{\pi}{2}\right)$ and $F\left(\frac{\pi}{2}\right) \preccurlyeq_S F(x)$. Therefore, $\frac{\pi}{2}$ is a solution of the problem wrt order relation $\preccurlyeq_S$.

Because $\frac{\pi}{2}$ is a solution of (12), it is also a weak maximal solution of (12). Similarly, $\frac{\pi}{4}$ is a weak maximal solution of (12).

Suppose that $\frac{\pi}{4}$ is strongly maximal solution of the problem. Then, $F(x) \preccurlyeq_S F\left(\frac{\pi}{4}\right)$ must be satisfied for all $x \in [0,2\pi]$. So,

For $x \in \left[0,\frac{\pi}{4}\right] \cup \left(\frac{5\pi}{4}, 2\pi\right]$,

$$F(x) \preccurlyeq_S F\left(\frac{\pi}{4}\right) \quad \Leftrightarrow \quad [\sin x, \cos x] \preccurlyeq_S \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$$
$$\Leftrightarrow \cos x \leq \frac{\sqrt{2}}{2}$$
$$\Leftrightarrow x \in \left\{\frac{\pi}{4}\right\} \cup \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right]$$

For $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]$,

$$F(x) \preccurlyeq_S F\left(\frac{\pi}{4}\right) \quad \Leftrightarrow \quad [\cos x, \sin x] \preccurlyeq_S \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$$
$$\Leftrightarrow \sin x \leq \frac{\sqrt{2}}{2}$$
$$\Leftrightarrow x \in \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$$

Then, $F(x) \preccurlyeq_S F\left(\frac{\pi}{4}\right)$ is satisfied for only $x \in \left\{\frac{\pi}{4}\right\} \cup \left[\frac{3\pi}{4}, \frac{7\pi}{4}\right]$. That is $F(x) \preccurlyeq_S F\left(\frac{\pi}{4}\right)$ is not satisfied for all $x \in [0,2\pi]$. Therefore, $\frac{\pi}{4}$ is not a strongly maximal solution.

Suppose that $\frac{\pi}{2}$ is a strongly maximal solution of the problem. Then, $F(x) \preccurlyeq_S F\left(\frac{\pi}{2}\right)$ must be satisfied for all $x \in [0,2\pi]$. So,

For $x \in \left[0,\frac{\pi}{4}\right] \cup \left(\frac{5\pi}{4}, 2\pi\right]$,

$$
\begin{aligned}
F(x) \preccurlyeq_s F\left(\frac{\pi}{2}\right) \quad &\Leftrightarrow \quad [\sin x, \cos x] \preccurlyeq_s \ [0,1] \\
&\Leftrightarrow \cos x \leq 0 \\
&\Leftrightarrow x \in \left(\frac{5\pi}{4}, \frac{3\pi}{2}\right]
\end{aligned}
$$

For $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]$,

$$
\begin{aligned}
F(x) \preccurlyeq_s F\left(\frac{\pi}{2}\right) \quad &\Leftrightarrow \quad [\cos x, \sin x] \preccurlyeq_s \ [0,1] \\
&\Leftrightarrow \sin x \leq 0 \\
&\Leftrightarrow x \in \left[\pi, \frac{5\pi}{4}\right]
\end{aligned}
$$

Then, $F(x) \preccurlyeq_s F\left(\frac{\pi}{2}\right)$ is satisfied for only $x \in \left(\frac{5\pi}{4}, \frac{3\pi}{2}\right] \cup \left[\pi, \frac{5\pi}{4}\right]$. That is $F(x)$ $\preccurlyeq_s F\left(\frac{\pi}{2}\right)$ is not satisfied for all $x \in [0,2\pi]$. Therefore, $\frac{\pi}{2}$ is not strongly maximal solution.

Finally, $\frac{\pi}{2}$ and $\frac{\pi}{2}$ are not strictly maximal solutions of the problem because of they are not strongly maximal solutions of the problem.

PART 4

SOME OPTIMALITY CRITERIA FOR INTERVAL OPTIMIZATION USING SUBDIFFERENTIALS

The aim of this chapter obtains some optimality conditions or optimality criteria for interval optimization problems using subdifferentials. These conditions allow us to have some knowledge about the solution(s) of interval optimization problems. With the help of optimality conditions, we can find the solutions of the problems or obtain the condition(s), which must be satisfied by their solution(s).

Subdifferential is a new subject for interval functions. The first subgradients and subdifferentials obtained by Karaman (2020) for interval-valued functions. Karaman (2020) obtained some optimality conditions for interval optimizations using subdifferentials. These conditions gave for only order relation $\preccurlyeq_s$. After that, Karaman (2021) defined the other subgradients and subdifferentials for interval functions and obtained some optimality criteria for interval optimization problems using order relations $\preccurlyeq_l$ and $\preccurlyeq_r$. Also, Ghosh vd. (2022) defined a generalization of subdifferentials defined given by Karaman (2020) and obtained some optimality conditions for interval optimization problems. These optimality conditions examined in the first part using examples.

4.1. s-SUBDIFFERENTIAL AND SOME OPTIMALITY CRITERIA FOR INTERVAL OPTIMIZATION USING s-SUBDIFFERENTIALS

Definition 4.1. [8] Let X be nonempty set and $F: X \to I(\mathbb{R})$ be an interval function. Then, linear operator $L: \mathbb{R}^n \to \mathbb{R}$ is called a s-subgradient of F at $x_0 \in X$ if

$$F(x) - F(x_0) \prec_s L(x - x_0)$$

for all $x \in X \,/\, \{x_0\}$. The set of all s-subgradients of F at x_0 is called the s-subdifferential of F at x_0, and denoted by $\partial^s F(x_0)$.

Then,

$$\partial^s F(x_0) = \{L: \mathbb{R}^n \to \mathbb{R} : \ F(x) - F(x_0) \prec_s L(x - x_0), \forall x \in X \ / \ \{x_0\}\}.$$

Definition 4.2. [8] Let $L: \mathbb{R}^n \to \mathbb{R}$ be a linear operator and $F: X \to I(\mathbb{R})$ be an interval function. L is called a weak s-subgradient of F at $x_0 \in X$ if

$$F(x) - F(x_0) \preccurlyeq_s L(x - x_0)$$

for all $x \in X$. The set of all s-weak subgradients of F at x_0 is called the weak s-subdifferential of F at x_0, and denoted by $\partial_w^s F(x_0)$.

Then,

$$\partial_w^s F(x_0) = \{L: \mathbb{R}^n \to \mathbb{R} : \ F(x) - F(x_0) \preccurlyeq_s L(x - x_0), \forall x \in X\}.$$

Let $F: X \to I(\mathbb{R})$ be an interval function. If the s-subdifferential of F at $x_0 \in X$ is nonempty set, then F is called s-subdifferentiable at x_0. Similarly, if the weak s-subdifferential of F at $x_0 \in X$ is nonempty set, then F is called weak s-subdifferentiable at x_0.

s-subdifferential and weak s-subdifferential have the following properties [8].

Let $F, G: X \to I(\mathbb{R})$ be interval functions and $x_0 \in X$. Then,

- $\partial^s F(x_0) \subseteq \partial_w^s F(x_0)$
- $\partial^s F(x_0)$ and $\partial_w^s F(x_0)$ are convex
- $\partial^s (kF)(x_0) = k\partial^s F(x_0)$ for all $k > 0$
- $\partial^s F(x_0)$ is closed
- $\partial^s G(x_0) + \partial^s F(x_0) \subseteq \partial^s (F + G)(x_0)$
- If there exists a constant K such that $K|x - x_0| \subsetneq F(x) - F(x_0)$ for all $x \in X \setminus \{x_0\}$, then $\partial^s F(x_0) \neq \emptyset$

Let's take into account the following interval optimization wrt $\preccurlyeq_s$:

$$(IVP) \begin{cases} \min(max) \, F(x) \\ \qquad x \in X \end{cases}$$

Theorem 4.1. [8] Let $x_0 \in X$ and (IVP) be given wrt order relation $\preccurlyeq_s$.

(i) If $0 \in \partial^s F(x_0)$, then x_0 is a maximal solution of (IVP)

(ii) If $0 \in \partial^s F(x_0)$, then x_0 is a weak maximal solution of (IVP)

(iii) If $0 \in \partial_w^s F(x_0)$, then x_0 is a maximal solution of (IVP)

4.2. mr-SUBDIFFERENTIAL AND SOME PROPERTIES

A new subdifferential and weak subdifferential will be defined using order relation $\preccurlyeq_{mr}$ to obtain new optimality criteria for interval optimization problem given by order relation $\preccurlyeq_{mr}$.

Definition 4.3. Let and $F: X \to I(\mathbb{R})$ be an interval function. Then, the linear operator $L: \mathbb{R}^n \to \mathbb{R}$ is called a mr-subgradient of F at $x_0 \in X$ if

$$F(x) -_g F(x_0) \prec_{mr} L(x - x_0)$$

for all $x \in X \,/\, \{x_0\}$. The set of all mr-subgradients of F at x_0 is called the mr-subdifferential of F at x_0, and denoted by $\partial^{mr} F(x_0)$.
Then,

$$\partial^{mr} F(x_0) = \{L: \mathbb{R}^n \to \mathbb{R} : \ F(x) -_g F(x_0) \prec_{mr} L(x - x_0), \forall x \in X \,/\, \{x_0\}\}.$$

Definition 4.4. Let $F: X \to I(\mathbb{R})$ be an interval function. Then, the linear operator $L: \mathbb{R}^n \to \mathbb{R}$ is called a weak mr-subgradient of F at $x_0 \in X$ if

$$F(x) -_g F(x_0) \preccurlyeq_{mr} L(x - x_0)$$

for all $x \in X$. The set of all weak mr-subgradients of F at x_0 is called the weak mr-subdifferential of F at x_0, and denoted by $\partial_w^{mr} F(x_0)$.
Then,

$$\partial_w^{mr} F(x_0) = \{L: \mathbb{R}^n \to \mathbb{R} : \ F(x) -_g F(x_0) \preccurlyeq_{mr} L(x - x_0), \forall x \in X\}.$$

Let $F: X \to I(\mathbb{R})$ be an interval function and $x_0 \in X$. If the mr-subdifferential of F at x_0 is nonempty set, that is $\partial^{mr} F(x_0) \neq \emptyset$, then F is called mr-subdifferentiable at x_0. Similarly, If the weak mr-subdifferential of F at x_0 is nonempty set, that is $\partial_w^{mr} F(x_0) \neq \emptyset$, then F is called weak mr-subdifferentiable at x_0.

mr-subdifferential and weak mr-subdifferential have the following properties.

Proposition 4.1. Let $F: X \to I(\mathbb{R})$ be mr-subdifferentiable interval function at $x_0 \in X$. Then, $\partial^{mr} F(x_0) \subseteq \partial_w^{mr} F(x_0)$.

Proof: Assume that L is a mr-subgradient of F at x_0, that is $L \in \partial^{mr} F(x_0)$. Then, we have

$$F(x) -_g F(x_0) \prec_{mr} L(x - x_0)$$

satisfies for all $x \in X \,/\, \{x_0\}$. Because $\prec_{mr}$ implies $\preccurlyeq_{mr}$, we get $F(x) -_g F(x_0) \preccurlyeq_{mr} L(x - x_0)$ for all $x \in X \,/\, \{x_0\}$. Since $F(x) -_g F(x_0) = 0$, we obtain

$$F(x) -_g F(x_0) \preccurlyeq_{mr} L(x - x_0)$$

for all $x \in X$. Then, L is a weak mr-subgradient of F at x_0, that is $L \in \partial_w^{mr} F(x_0)$.

Proposition 4.2. Let X be nonempty set and $F: X \to I(\mathbb{R})$ be an interval function. Then, $\partial^{mr} F(x_0)$ and $\partial_w^{mr} F(x_0)$ are convex sets.

Proof: Let $L_1, L_2 \in \partial^{mr} F(x_0)$ and $\lambda \in (0,1)$. Since $L_1, L_2 \in \partial^{mr} F(x_0)$, we have

$$F(x) -_g F(x_0) \prec_{mr} L_1(x - x_0)$$

and

$$F(x) -_g F(x_0) \prec_{mr} L_2(x - x_0)$$

for all $x \in X \,/\, \{x_0\}$. From Proposition 2.2, we have

$$\lambda\left(F(x) -_g F(x_0)\right) \prec_{mr} \lambda L_1(x - x_0)$$

and

$$(1-\lambda)\left(F(x) -_g F(x_0)\right) \prec_{mr} (1-\lambda) L_2(x - x_0)$$

for all $x \in X \,/\, \{x_0\}$. By Proposition 2.3, we get

$$\lambda\left(F(x) -_g F(x_0)\right) + (1-\lambda)\left(F(x) -_g F(x_0)\right) \prec_{mr} \lambda L_1(x - x_0) + (1-\lambda) L_2(x - x_0)$$

for all $x \in X \,/\, \{x_0\}$.

Then,

$$F(x) -_g F(x_0) \prec_{mr} \lambda L_1(x - x_0) + (1-\lambda) L_2(x - x_0),$$

for all $x \in X \,/\, \{x_0\}$ from Proposition 2.1. Then, $F(x) -_g F(x_0) \prec_{mr} = (\lambda L_1 + (1-\lambda)L_2(x - x_0)$ for all $x \in X \,/\, \{x_0\}$, implies that $(\lambda L_1 + (1-\lambda)L_2 \in \partial^{mr} F(x_0)$. Therefore, $\partial^{mr} F(x_0)$ is a convex set. Similarly, the convexity of $\partial_w^{mr} F(x_0)$ can be proved.

Proposition 4.3. Let $F: X \to I(\mathbb{R})$ be an mr-subdifferentiable interval function at $x_0 \in X$ and k be positive number. Then, $\partial^{mr}(kF)(x_0) = k\partial^{mr} F(x_0)$.

Proof: Let $k > 0$. Then,

$$\begin{aligned}
\partial^{mr}(kF)(x_0) &= \{L: \mathbb{R}^n \to \mathbb{R} : \ kF(x) -_g kF(x_0) \prec_{mr} L(x - x_0), \forall x \in X \,/\, \{x_0\}\} \\
&= \left\{L: \mathbb{R}^n \to \mathbb{R} : \ k\left(F(x) -_g F(x_0)\right) \prec_{mr} L(x - x_0), \forall x \in X \,/\, \{x_0\}\right\} \\
&= \left\{L: \mathbb{R}^n \to \mathbb{R} : \ F(x) -_g F(x_0) \prec_{mr} \frac{L}{k}(x - x_0), \forall x \in X \,/\, \{x_0\}\right\} \\
&= \{kL: \mathbb{R}^n \to \mathbb{R} : \ F(x) -_g F(x_0) \prec_{mr} L(x - x_0), \forall x \in X \,/\, \{x_0\}\} \\
&= k\{L: \mathbb{R}^n \to \mathbb{R} : \ F(x) -_g F(x_0) \prec_{mr} L(x - x_0), \forall x \in X \,/\, \{x_0\}\} \\
&= k\partial^{mr} F(x_0)
\end{aligned}$$

4.3. SOME OPTIMALITY CRITERIA FOR INTERVAL OPTIMIZATION USING mr-SUBDIFFERENTIAL

Let's again take into account the following interval optimization problem:

$$(IVP) \begin{cases} \max F(x) \\ x \in X \end{cases}$$

where $F: X \to I(\mathbb{R})$ is interval function. We will obtain some optimality criteria for (IVP) wrt $\preccurlyeq_{mr}$.

Theorem 4.2. Let $x_0 \in X$ and (IVP) be given wrt order relation $\preccurlyeq_{mr}$. If x_0 is a strongly solution of (IVP), then $0 \in \partial_w^{mr} F(x_0)$.

Proof: Let x_0 be a strongly solution of (IVP). Then, we $F(x) \preccurlyeq_{mr} F(x_0)$ for all $x \in X$. From Proposition 2.6, we get

$$F(x) -_g F(x_0) \preccurlyeq_{mr} 0$$

for all $x \in X$. Therefore, $0 \in \partial_w^{mr} F(x_0)$.

Example 4.2. Let interval function $F:[0,2\pi] \to I(\mathbb{R})$ be defined as

$$F(x) = \begin{cases} [\sin(x), \cos(x)] & ; 0 \le x \le \frac{\pi}{4} \ or \ \frac{5\pi}{4} < x \le 2\pi \\ [\cos(x), \sin(x)] & ; \frac{\pi}{4} < x \le \frac{5\pi}{4} \end{cases}$$

Consider the following interval optimization

$$\begin{cases} \max F(x) \\ x \in [0,2\pi]. \end{cases}$$

We know that $\frac{\pi}{4}$ is a strongly maximal solution of the problem. We should find that $0 \in \partial_w^{mr} F\left(\frac{\pi}{4}\right)$ from Theorem 4.2. Let's calculate the weak subdifferential of F at $\frac{\pi}{4}$.

$$\partial_w^{mr} F\left(\frac{\pi}{4}\right) = \left\{L: \mathbb{R} \to \mathbb{R} : F(x) -_g F\left(\frac{\pi}{4}\right) \preccurlyeq_{mr} L\left(x - \frac{\pi}{4}\right), x \in [0,2\pi]\right\}$$

$$= \left\{L: \mathbb{R} \to \mathbb{R} : F(x) -_g F\left(\frac{\pi}{4}\right) \preccurlyeq_{mr} L\left(x - \frac{\pi}{4}\right), x \in \left[0, \frac{\pi}{4}\right] \cup \left(\frac{5\pi}{4}, 2\pi\right]\right\}$$

$$\cap \left\{L: \mathbb{R} \to \mathbb{R} : F(x) -_g F\left(\frac{\pi}{4}\right) \preccurlyeq_{mr} L\left(x - \frac{\pi}{4}\right), x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]\right\}$$

$$= \left\{t \in \mathbb{R} : [\sin x, \cos x] -_g \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] \preccurlyeq_{mr} t\left(x - \frac{\pi}{4}\right), x \in \left[0, \frac{\pi}{4}\right] \cup \left(\frac{5\pi}{4}, 2\pi\right]\right\}$$

$$\cap \left\{t \in \mathbb{R} : [\cos x, \sin x] -_g \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] \preccurlyeq_{mr} t\left(x - \frac{\pi}{4}\right), x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]\right\}$$

$$= \left\{t \in \mathbb{R} : \left[\sin x - \frac{\sqrt{2}}{2}, \cos x - \frac{\sqrt{2}}{2}\right] \preccurlyeq_{mr} t\left(x - \frac{\pi}{4}\right), x \in \left[0, \frac{\pi}{4}\right] \cup \left(\frac{5\pi}{4}, 2\pi\right]\right\}$$

$$\cap \left\{t \in \mathbb{R} : \left[\cos x - \frac{\sqrt{2}}{2}, \sin x - \frac{\sqrt{2}}{2}\right] \preccurlyeq_{mr} t\left(x - \frac{\pi}{4}\right), x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]\right\}$$

$$= \left\{t \in \mathbb{R} : \left[\sin x - \frac{\sqrt{2}}{2}, \cos x - \frac{\sqrt{2}}{2}\right] \preccurlyeq_{mr} t\left(x - \frac{\pi}{4}\right), x \in \left[0, \frac{\pi}{4}\right]\right\}$$

$$\cap \left\{t \in \mathbb{R} : \left[\sin x - \frac{\sqrt{2}}{2}, \cos x - \frac{\sqrt{2}}{2}\right] \preccurlyeq_{mr} t\left(x - \frac{\pi}{4}\right), x \in \left(\frac{5\pi}{4}, 2\pi\right]\right\}$$

$$\cap \left\{t \in \mathbb{R} : \left[\cos x - \frac{\sqrt{2}}{2}, \sin x - \frac{\sqrt{2}}{2}\right] \preccurlyeq_{mr} t\left(x - \frac{\pi}{4}\right), x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]\right\}$$

$$= \left\{t \in \mathbb{R} : \frac{\sin x + \cos x - \sqrt{2}}{2} \leq t\left(x - \frac{\pi}{4}\right) \; and \; 0 \leq \frac{\cos x - \sin x}{2}, x \in \left[0, \frac{\pi}{4}\right]\right\}$$

$$\cap \left\{t \in \mathbb{R} : \frac{\sin x + \cos x - \sqrt{2}}{2} \leq t\left(x - \frac{\pi}{4}\right) \; and \; 0 \leq \frac{\cos x - \sin x}{2}, x \in \left(\frac{5\pi}{4}, 2\pi\right]\right\}$$

$$\cap \left\{t \in \mathbb{R} : \frac{\sin x + \cos x - \sqrt{2}}{2} \leq t\left(x - \frac{\pi}{4}\right) \; and \; 0 \leq \frac{\sin x - \cos x}{2}, x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]\right\}$$

$$= \left\{t \in \mathbb{R} : \frac{\sin x + \cos x - \sqrt{2}}{2\left(x - \frac{\pi}{4}\right)} \geq t \, , x \in \left[0, \frac{\pi}{4}\right]\right\}$$

$$\cap \left\{t \in \mathbb{R} : \frac{\sin x + \cos x - \sqrt{2}}{2\left(x - \frac{\pi}{4}\right)} \leq t, x \in \left(\frac{5\pi}{4}, 2\pi\right]\right\}$$

$$\cap \left\{t \in \mathbb{R} : \frac{\sin x + \cos x - \sqrt{2}}{2\left(x - \frac{\pi}{4}\right)} \leq t, x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]\right\}$$

$$= \{t \in \mathbb{R} : t \leq 0\} \cap \left\{t \in \mathbb{R} : \frac{2 - 2\sqrt{2}}{7\pi} \leq t\right\} \cap \{t \in \mathbb{R} : 0 \leq t\} = \{0\}.$$

We can get the following result because all strictly maximal solution of (IVP) is a strongly maximal solution of (IVP).

Conclusion 4.1. Let $x_0 \in X$ and (IVP) be given wrt order relation $\preccurlyeq_{mr}$. If x_0 is a strictly maximal solution of (IVP), then $0 \in \partial_w^{mr} F(x_0)$.

Theorem 4.3. Let $x_0 \in X$ and (IVP) be given wrt order relation $\preccurlyeq_{mr}$. If x_0 is a strictly maximal solution of (IVP), then $0 \in \partial^{mr} F(x_0)$.

Proof: Let x_0 be a strictly maximal solution of (IVP). Then, we have $F(x) \prec_{mr} F(x_0)$ for all $x \in X \,/\, \{x_0\}$. From Proposition 2.6, we get

$$F(x) -_g F(x_0) \prec_{mr} 0$$

for all $x \in X \,/\, \{x_0\}$. Therefore, $0 \in \partial^{mr} F(x_0)$.

Example 4.3. Let interval function $F: [0,2\pi] \to I(\mathbb{R})$ be defined as

$$F(x) = \begin{cases} [\sin(x), \cos(x)] & ; 0 \leq x \leq \frac{\pi}{4} \ or \ \frac{5\pi}{4} < x \leq 2\pi \\ [\cos(x), \sin(x)] & ; \frac{\pi}{4} < x \leq \frac{5\pi}{4} \end{cases}$$

Consider the following interval optimization

$$\begin{cases} \max F(x) \\ x \in [0,2\pi]. \end{cases}$$

We know that $\frac{\pi}{4}$ is a strictly maximal solution of the problem. We must find that $0 \in \partial^{mr} F\left(\frac{\pi}{4}\right)$ from Theorem 4.3. Let's calculate the subdifferential of F at $\frac{\pi}{4}$.

$$\partial^{mr} F\left(\frac{\pi}{4}\right) = \left\{L: \mathbb{R} \to \mathbb{R} : F(x) -_g F\left(\frac{\pi}{4}\right) \prec_{mr} L\left(x - \frac{\pi}{4}\right), x \in [0,2\pi] / \left\{\frac{\pi}{4}\right\}\right\}$$

$$= \left\{L: \mathbb{R} \to \mathbb{R} : F(x) -_g F\left(\frac{\pi}{4}\right) \prec_{mr} L\left(x - \frac{\pi}{4}\right), x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]\right\}$$

$$\cap \left\{L: \mathbb{R} \to \mathbb{R} : F(x) -_g F\left(\frac{\pi}{4}\right) \prec_{mr} L\left(x - \frac{\pi}{4}\right), x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]\right\}$$

$$= \left\{t \in \mathbb{R} : [\sin x, \cos x] -_g \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] \prec_{mr} t\left(x - \frac{\pi}{4}\right), x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]\right\}$$

$$\cap \left\{t \in \mathbb{R} : [\cos x, \sin x] -_g \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right] \prec_{mr} t\left(x - \frac{\pi}{4}\right), x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]\right\}$$

$$= \left\{t \in \mathbb{R} : \left[\sin x - \frac{\sqrt{2}}{2}, \cos x - \frac{\sqrt{2}}{2}\right] \prec_{mr} t\left(x - \frac{\pi}{4}\right), x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]\right\}$$

$$\cap \left\{t \in \mathbb{R} : \left[\cos x - \frac{\sqrt{2}}{2}, \sin x - \frac{\sqrt{2}}{2}\right] \prec_{mr} t\left(x - \frac{\pi}{4}\right), x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]\right\}$$

$$= \left\{t \in \mathbb{R} : \left[\sin x - \frac{\sqrt{2}}{2}, \cos x - \frac{\sqrt{2}}{2}\right] \prec_{mr} t\left(x - \frac{\pi}{4}\right), x \in \left[0, \frac{\pi}{4}\right)\right\}$$

$$\cap \left\{t \in \mathbb{R} : \left[\sin x - \frac{\sqrt{2}}{2}, \cos x - \frac{\sqrt{2}}{2}\right] \prec_{mr} t\left(x - \frac{\pi}{4}\right), x \in \left(\frac{5\pi}{4}, 2\pi\right]\right\}$$

$$\cap \left\{t \in \mathbb{R} : \left[\cos x - \frac{\sqrt{2}}{2}, \sin x - \frac{\sqrt{2}}{2}\right] \prec_{mr} t\left(x - \frac{\pi}{4}\right), x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right]\right\}$$

$$= \left\{ t \in \mathbb{R} : \frac{\sin x + \cos x - \sqrt{2}}{2} \leq t\left(x - \frac{\pi}{4}\right) \ and \ 0 \leq \frac{\cos x - \sin x}{2}, x \in \left[0, \frac{\pi}{4}\right) \right\}$$

$$\cap \left\{ t \in \mathbb{R} : \frac{\sin x + \cos x - \sqrt{2}}{2} \leq t\left(x - \frac{\pi}{4}\right) \ and \ 0 \leq \frac{\cos x - \sin x}{2}, x \in \left(\frac{5\pi}{4}, 2\pi\right] \right\}$$

$$\cap \left\{ t \in \mathbb{R} : \frac{\sin x + \cos x - \sqrt{2}}{2} \leq t\left(x - \frac{\pi}{4}\right) \ and \ 0 \leq \frac{\sin x - \cos x}{2}, x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right] \right\}$$

$$= \left\{ t \in \mathbb{R} : \frac{\sin x + \cos x - \sqrt{2}}{2\left(x - \frac{\pi}{4}\right)} \geq t, x \in \left[0, \frac{\pi}{4}\right) \right\}$$

$$\cap \left\{ t \in \mathbb{R} : \frac{\sin x + \cos x - \sqrt{2}}{2\left(x - \frac{\pi}{4}\right)} \leq t, x \in \left(\frac{5\pi}{4}, 2\pi\right] \right\}$$

$$\cap \left\{ t \in \mathbb{R} : \frac{\sin x + \cos x - \sqrt{2}}{2\left(x - \frac{\pi}{4}\right)} \leq t, x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right] \right\}$$

$$= \{t \in \mathbb{R} : t \leq 0\} \cap \left\{ t \in \mathbb{R} : \frac{2 - 2\sqrt{2}}{7\pi} \leq t \right\} \cap \{t \in \mathbb{R} : 0 \leq t\} = \{0\}.$$

Theorem 4.4. Let $x_0 \in X$ and (IVP) be given wrt order relation $\preccurlyeq_{mr}$. Then, if x_0 is a minimal solution of (IVP), then $0 \notin \partial^{mr} F(x_0)$.

Proof: Assume that x_0 is a minimal solution of (IVP). Then, there is no an $x \in X$ such that $F(x) \preccurlyeq_{mr} F(x_0)$ and $F(x) \neq F(x_0)$. From Proposition 2.6, we know that there is no an $x \in X$ such that $F(x) -_g F(x_0) \preccurlyeq_{mr} 0$ and $F(x) \neq F(x_0)$. Then, $0 \notin \partial_w^{mr} F(x_0)$.

Theorem 4.5. Let $x_0 \in X$ and (IVP) be given wrt order relation $\preccurlyeq_{mr}$. Then, if x_0 is a weak minimal solution of (IVP), then $0 \notin \partial_w^{mr} F(x_0)$.

Proof: Assume that x_0 is a weak minimal solution of (IVP). Then, there is no an $x \in X$ such that $F(x) \prec_{mr} F(x_0)$. From Proposition 2.6, we know that there is no an $x \in X$ such that $F(x) -_g F(x_0) \prec_{mr} 0$. Then, $0 \notin \partial^{mr} F(x_0)$.

PART 5

CONCLUSION

In this thesis, interval numbers, interval functions, interval optimization and solution of them are considered. The importance of intervals and some applications in literature are given in the first part. The construction of interval-valued numbers and interval functions are given in the second part. Some notations and definitions on intervals and interval functions are explained using examples. In the third part, we interval optimization problems and solutions are presented. Two interval optimizations was solved using different order relations. We show that the solutions of interval optimization can change depends to order relations. Some optimality conitions for interval optimization are recalled and obtained in the last part. Optimality conditions are important in the optimization theory because they give some conditions that solutions must be satisfied them. We defined a subdifferential and examined some properties of them. Later, we obtained some optimality conditions for interval optimization wrt order relation $\preccurlyeq_{mr}$ using this subdifferential. Obtained theoretical results were proved and them examined on examples.

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RESUME

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